First-Order Logic

Part One

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

$$Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

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$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

These blue terms are called *constant symbols*. Unlike propositional variables, they refer to *objects*, not propositions.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

The red things that look like function calls are called *predicates*. Predicates take objects as arguments and evaluate to true or false.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

$$Cute(Quokka)$$

$$ArgueIncessantly(Democrats, Republicans)$$

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$Cute(a) \to Dikdik(a) \lor Kitty(a) \lor Puppy(a)$$

$$Succeeds(You) \leftrightarrow Practices(You)$$

$$x < 8 \to x < 137$$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this way.

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as $\to$ and $\neg$ are.
- Examples:

$$TomMarvoloRiddle = LordVoldemort$$

$$MorningStar = EveningStar$$

- Equality can only be applied to ***objects***; to state that two ***propositions*** are equal, use $\leftrightarrow$.

Let's see some more examples.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

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These purple terms are *functions*. Functions take objects as input and produce objects as output.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

$$\mathit{FavoriteMovieOf}(\mathit{You}) \neq \mathit{FavoriteMovieOf}(\mathit{Date}) \land$$
$$\mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{You})) = \mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{Date}))$$

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

$$ColorOf(Money)$$

$$MedianOf(x, y, z)$$

$$x + y$$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

Objects and Predicates

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.
- You cannot apply connectives to objects:

⚠ $\mathit{Venus} \to \mathit{TheSun}$ ⚠

- You cannot apply functions to propositions:

⚠ $\mathit{StarOf}(\mathit{IsRed}(\mathit{Sun}) \land \mathit{IsGreen}(\mathit{Mars}))$ ⚠

- Ever get confused? *Just ask!*

Some muggle is intelligent.

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

The Existential Quantifier

- A statement of the form

  $$\exists x.\ \textit{some-formula}$$

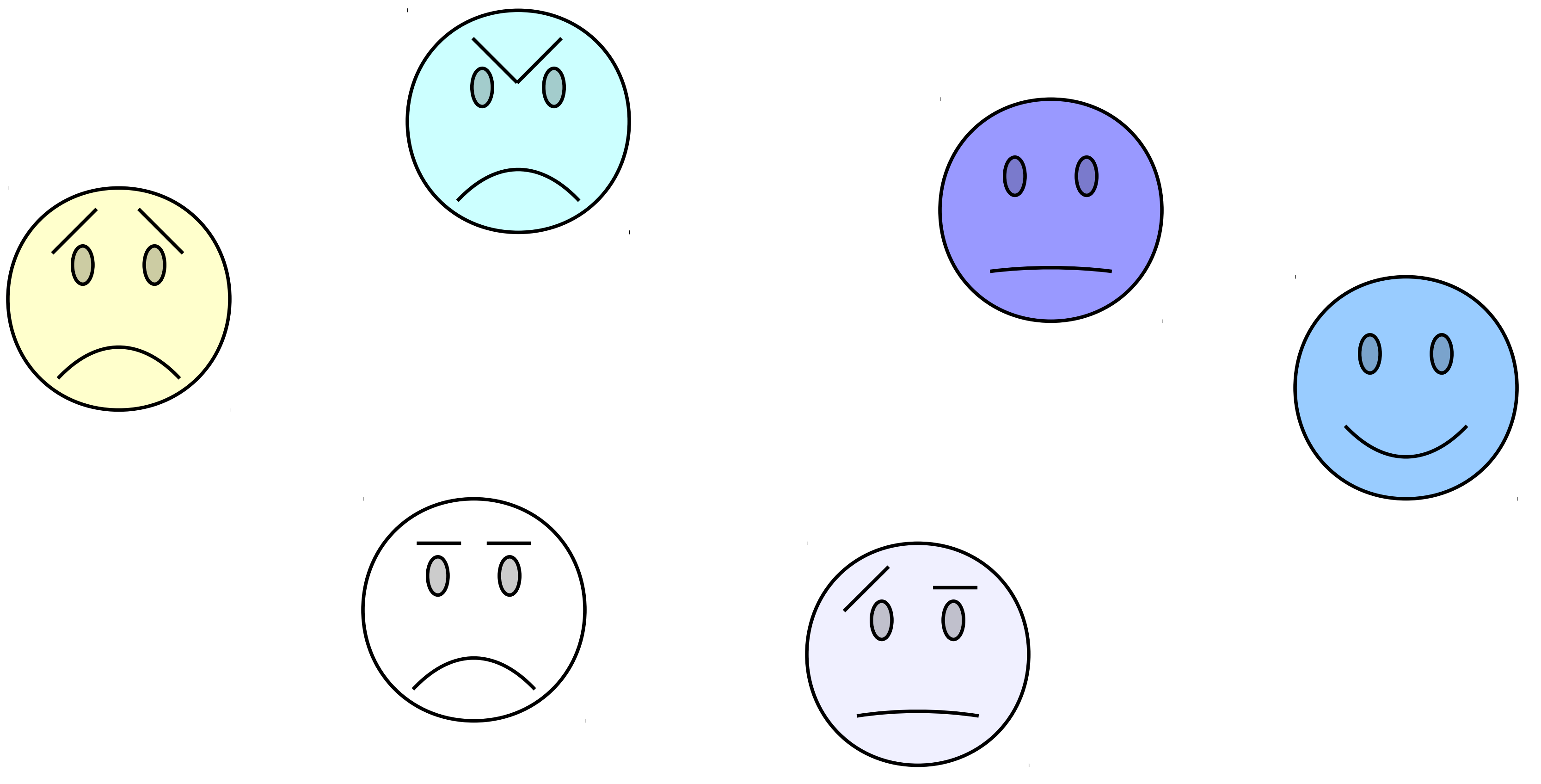
  is true if, for *some* choice of x, the statement ***some-formula*** is true when that x is plugged into it.
- Examples:

  $\exists x.\ (Even(x) \land Prime(x))$

  $\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$

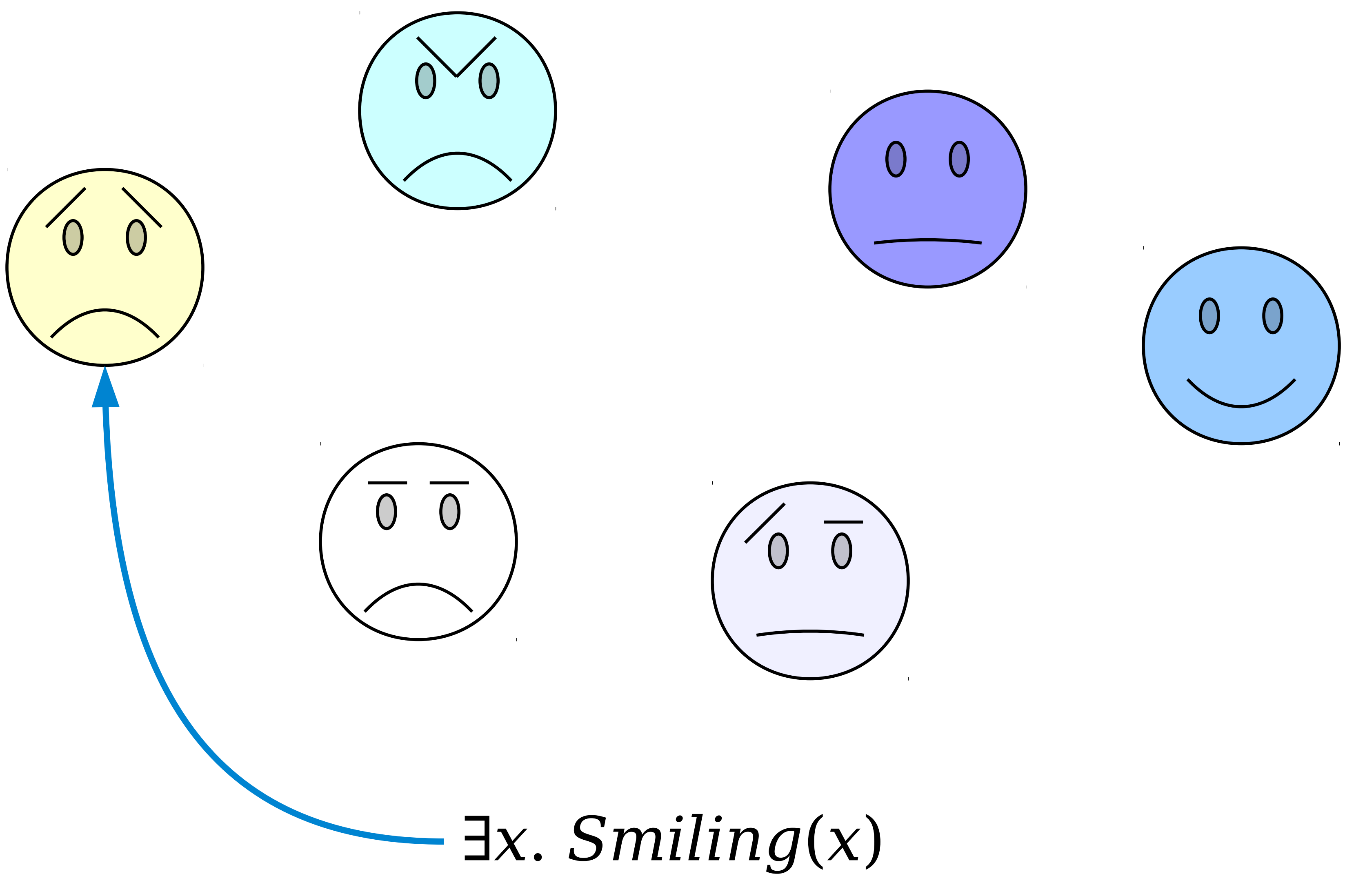
  $(\exists w.\ Will(w)) \to (\exists x.\ Way(x))$

The Existential Quantifier

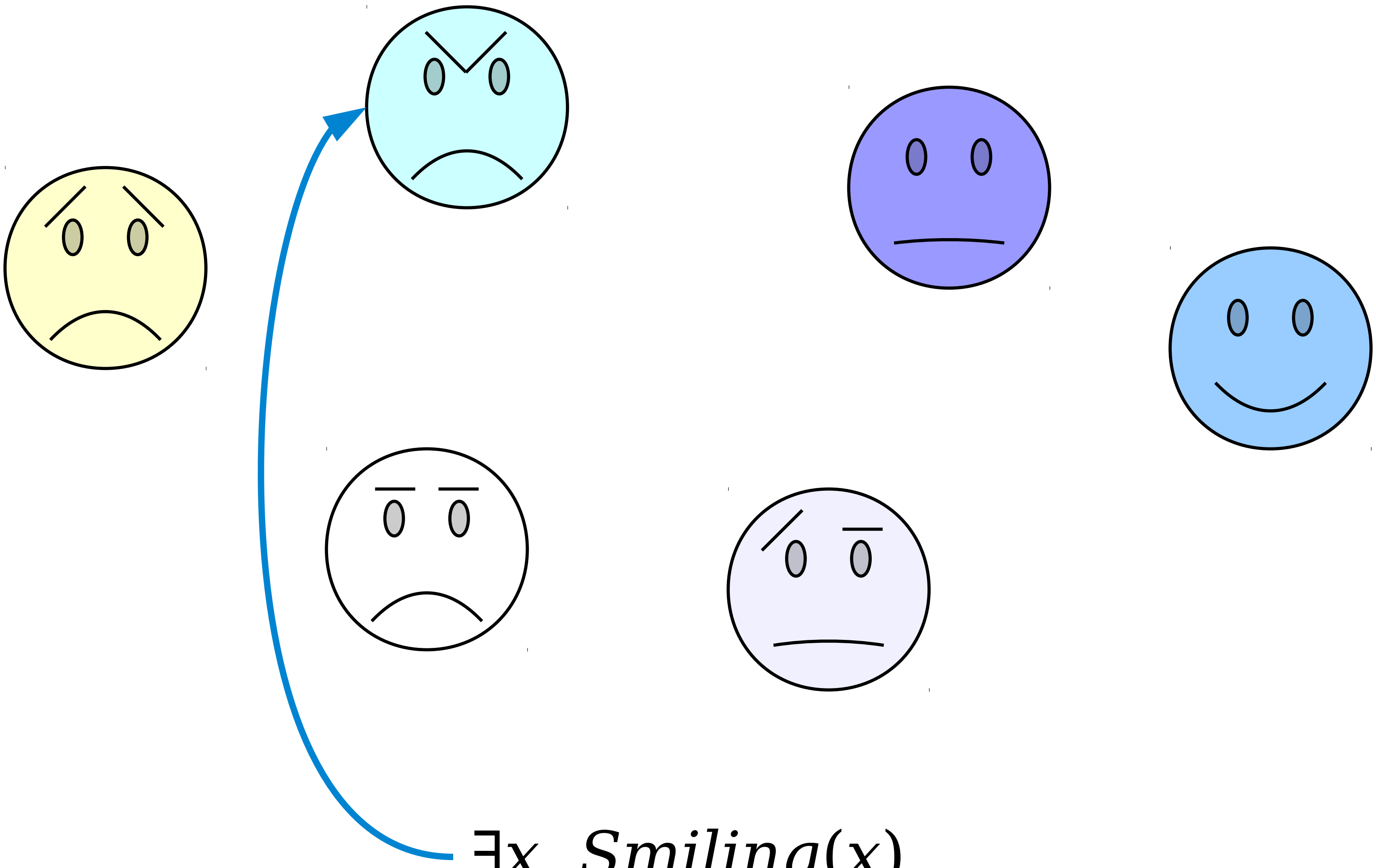


$$\exists x.\ Smiling(x)$$

The Existential Quantifier

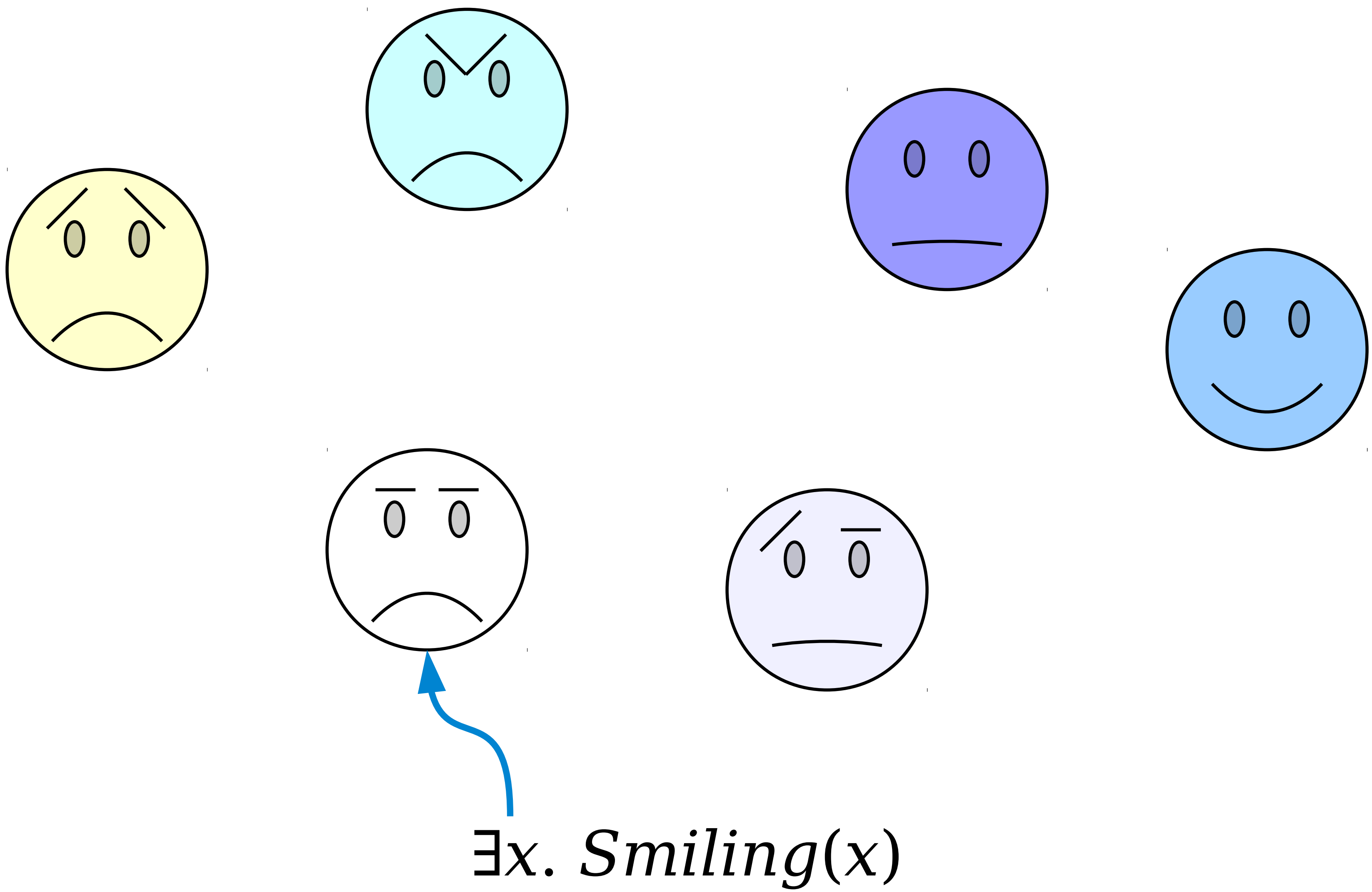


The Existential Quantifier



$\exists x.\ Smiling(x)$

The Existential Quantifier



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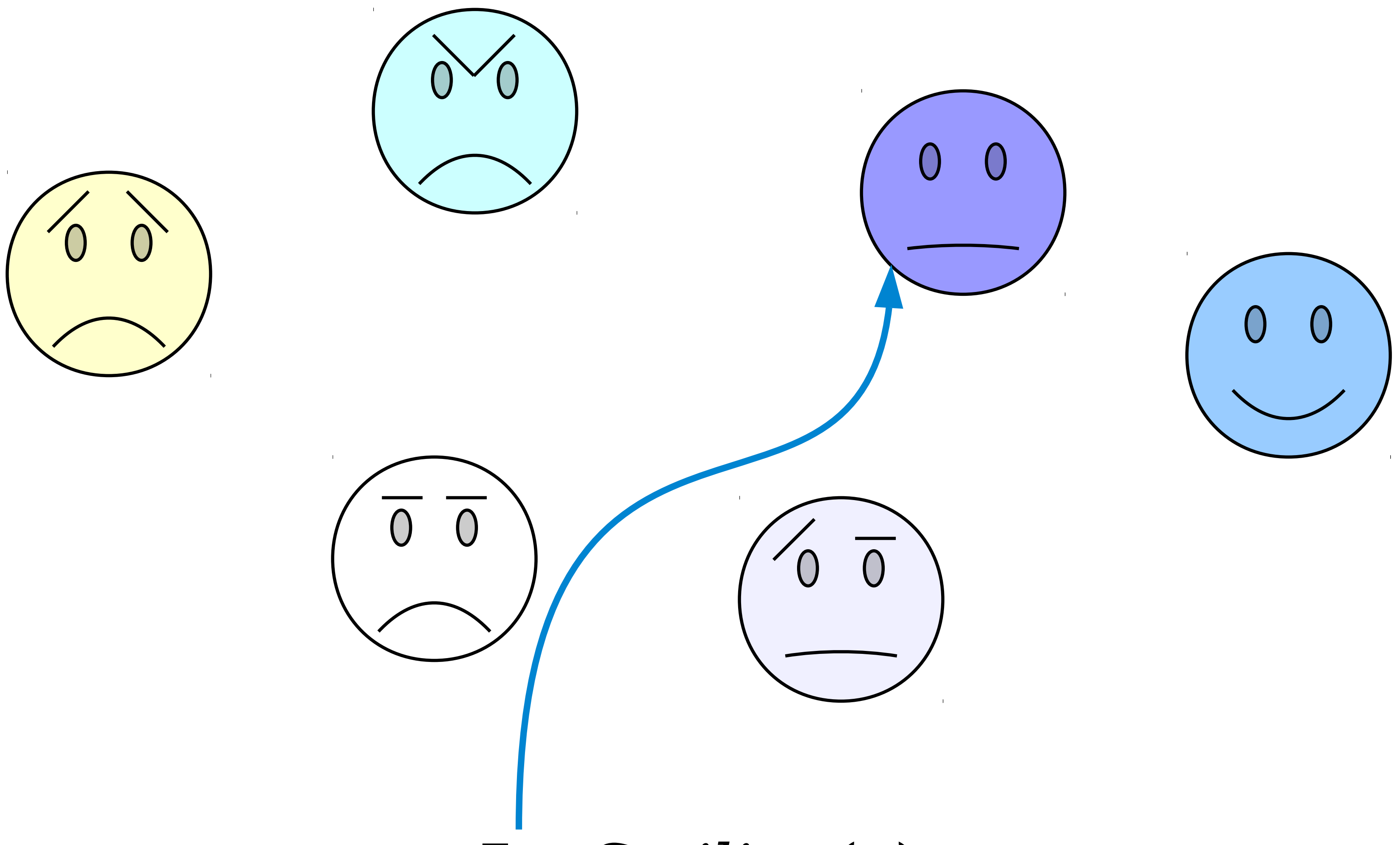

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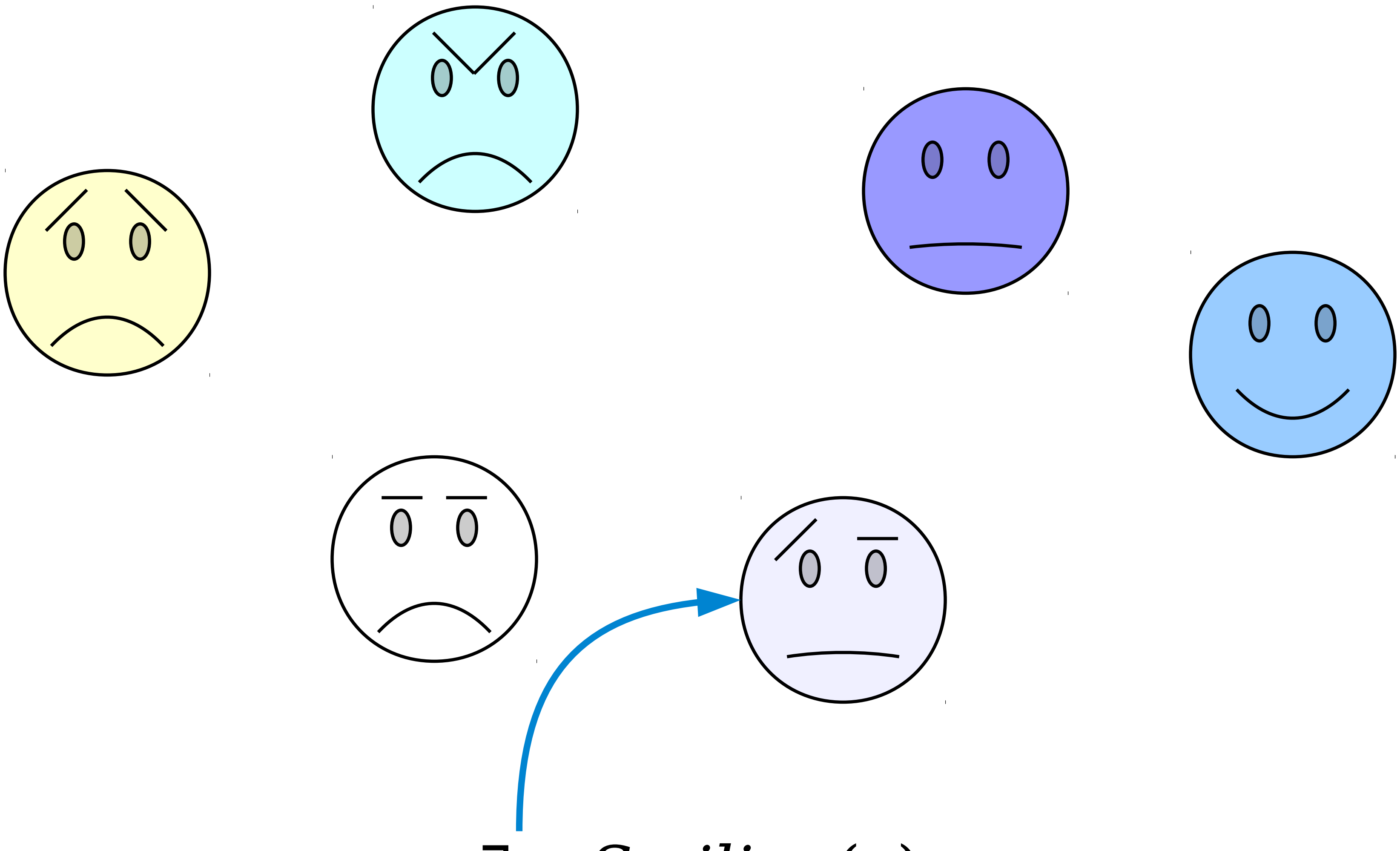

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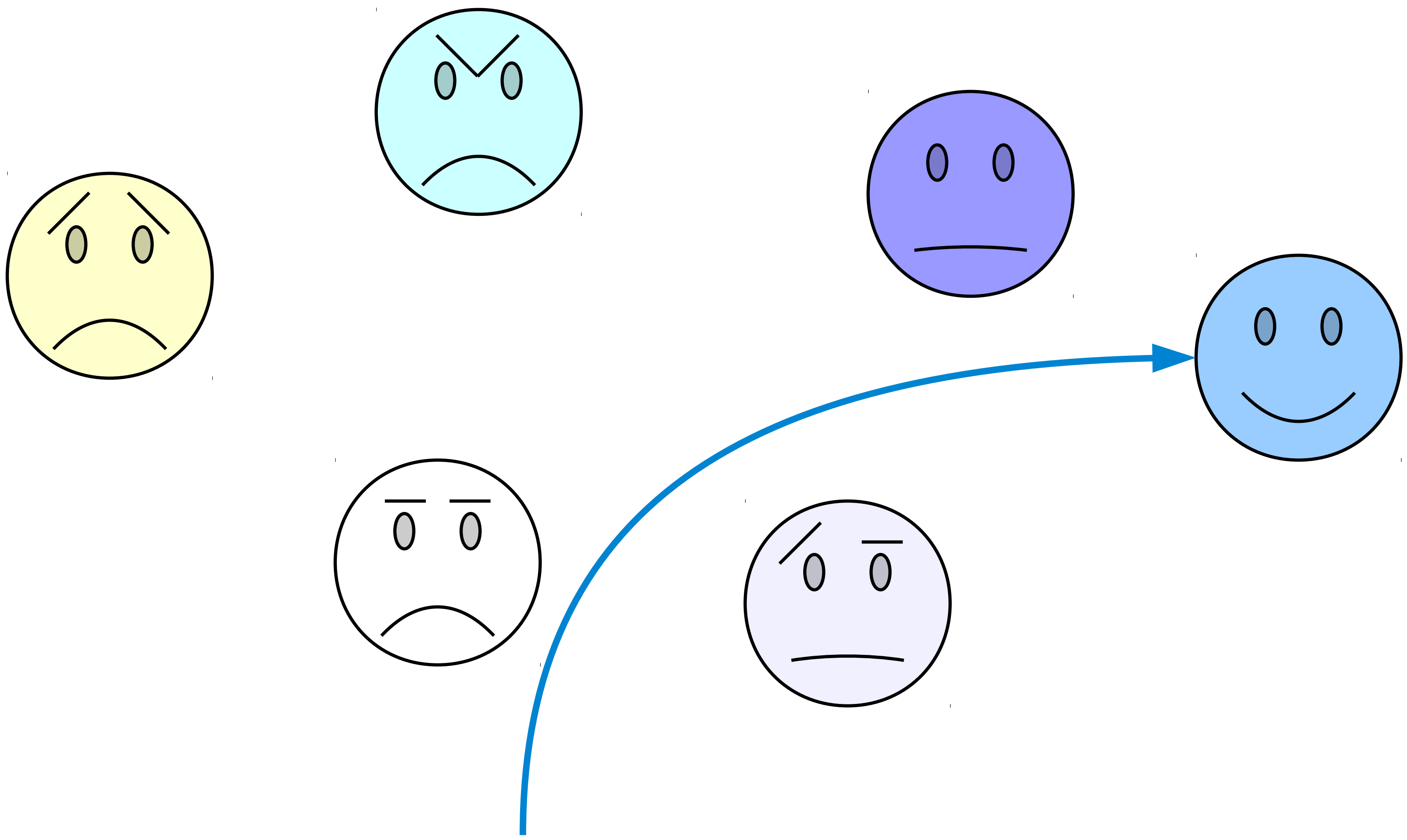

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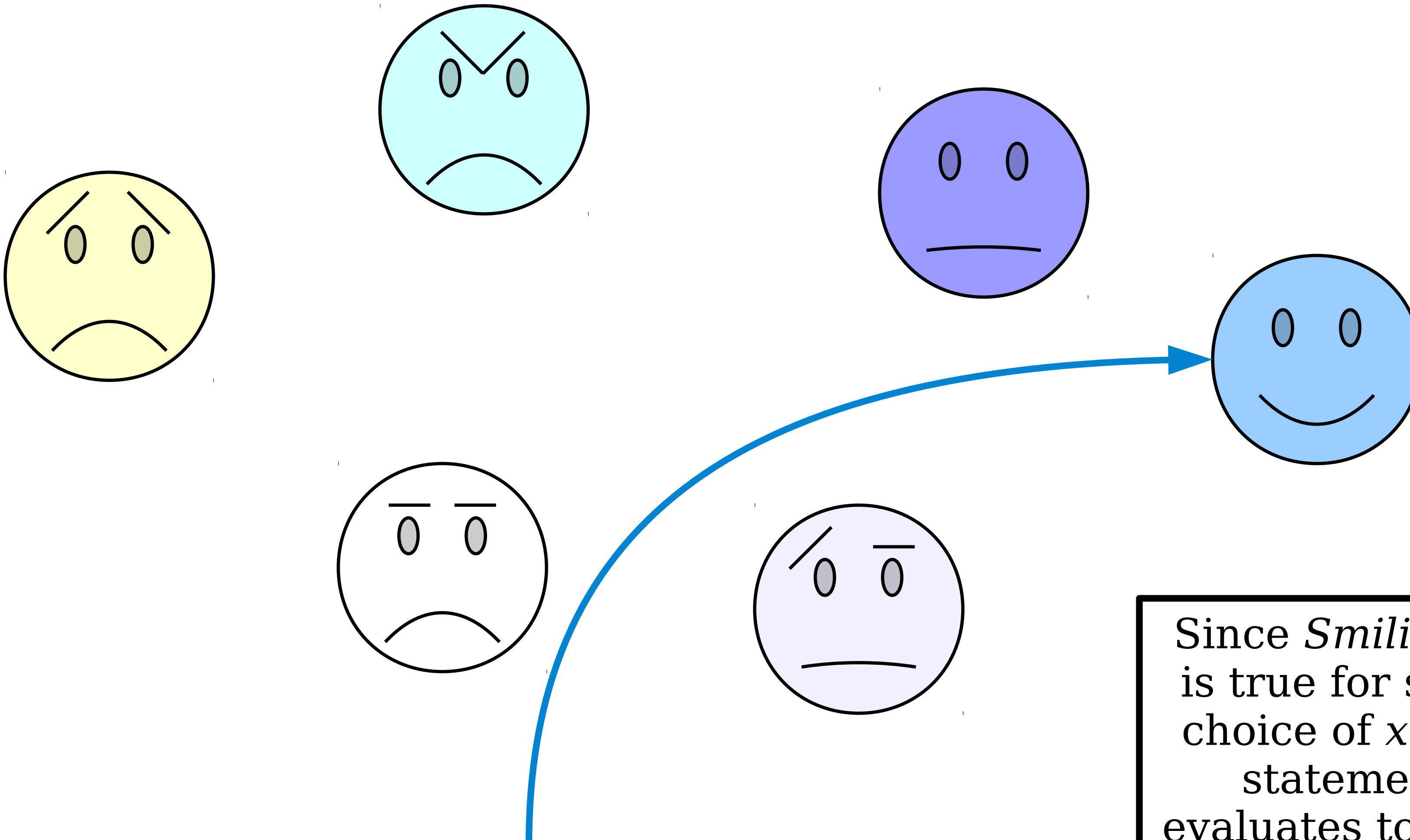
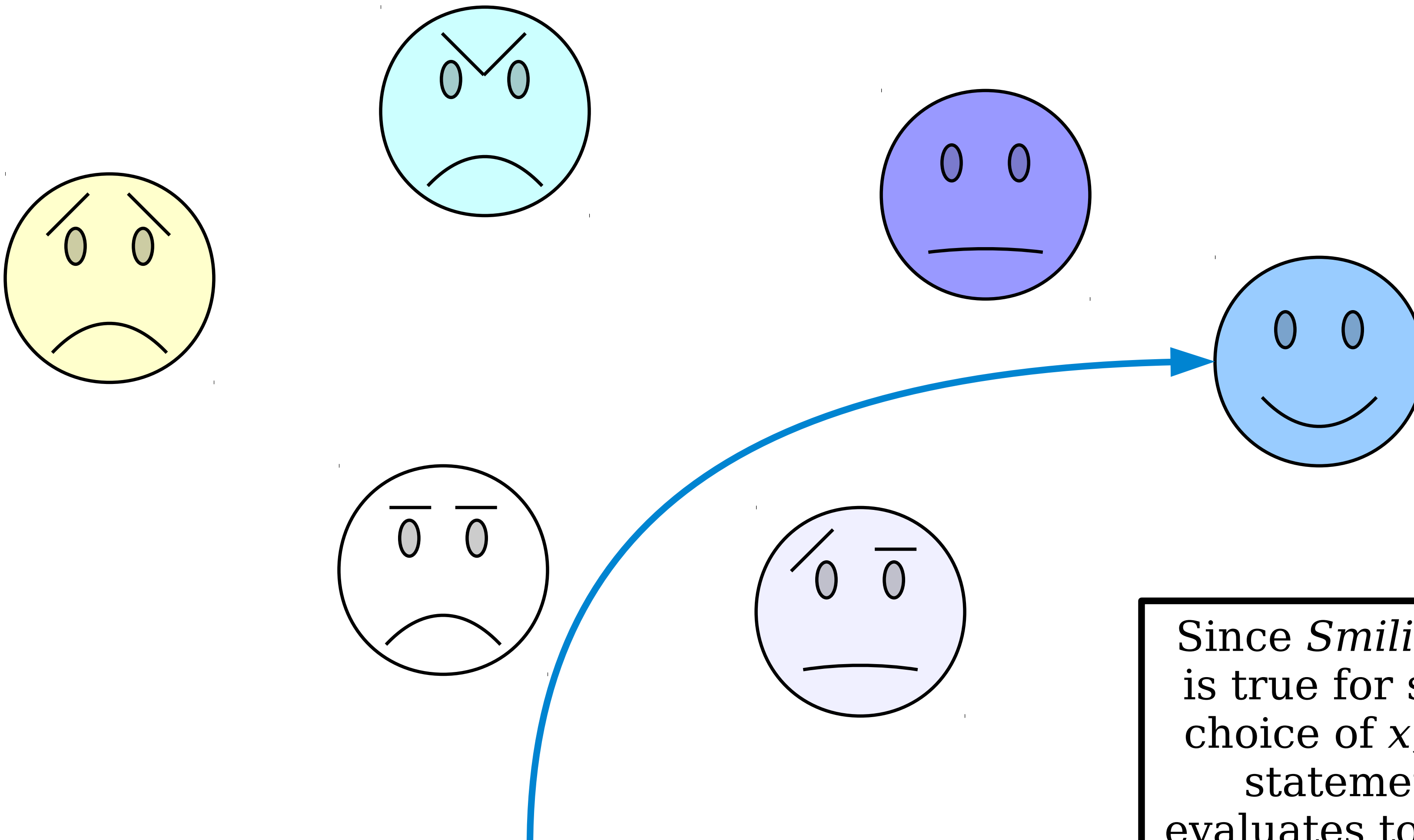
The Existential Quantifier



$\exists x.\ Smiling(x)$

Since *Smiling*(*x*) is true for some choice of *x*, this statement evaluates to true.

The Existential Quantifier



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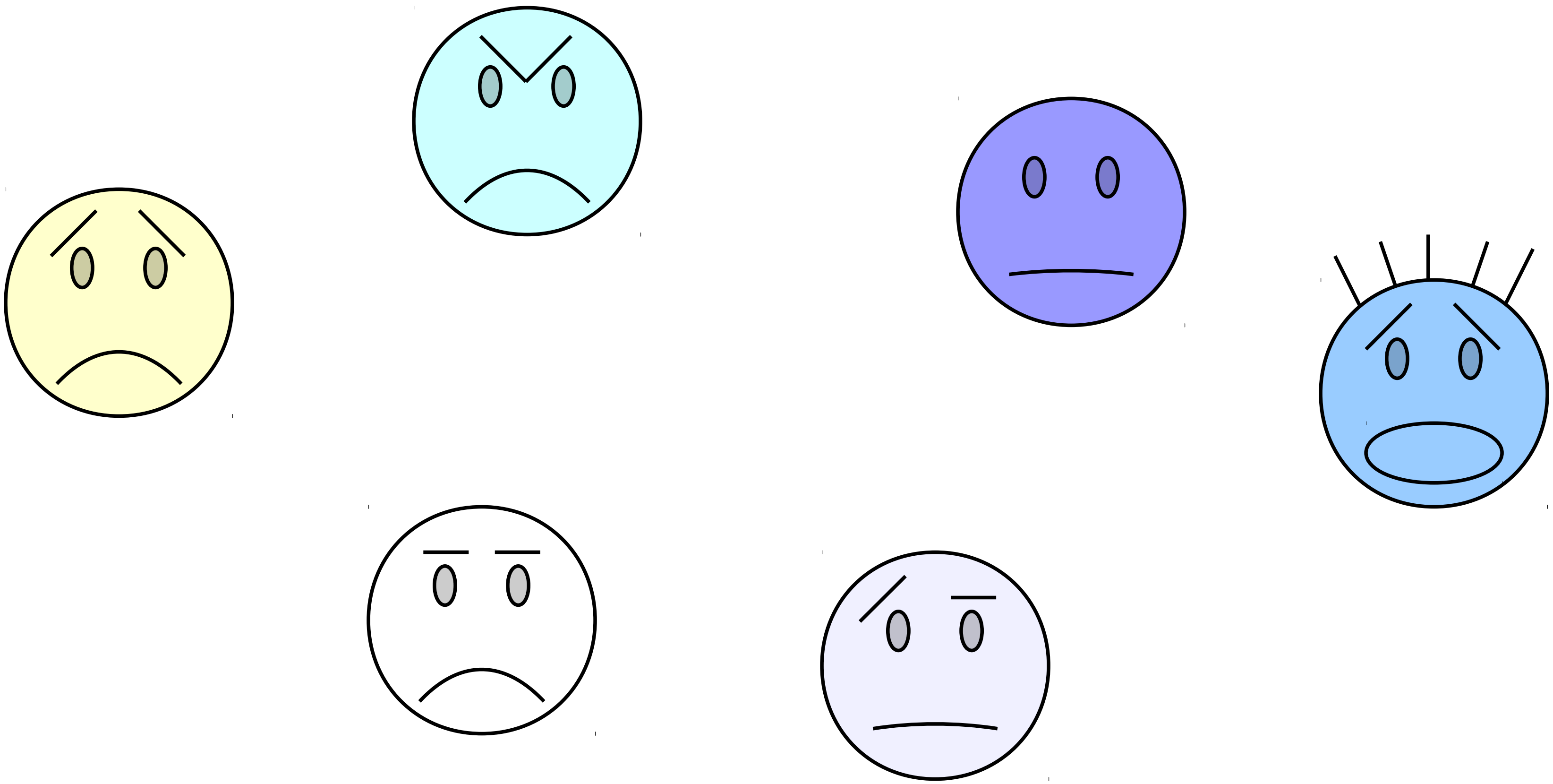

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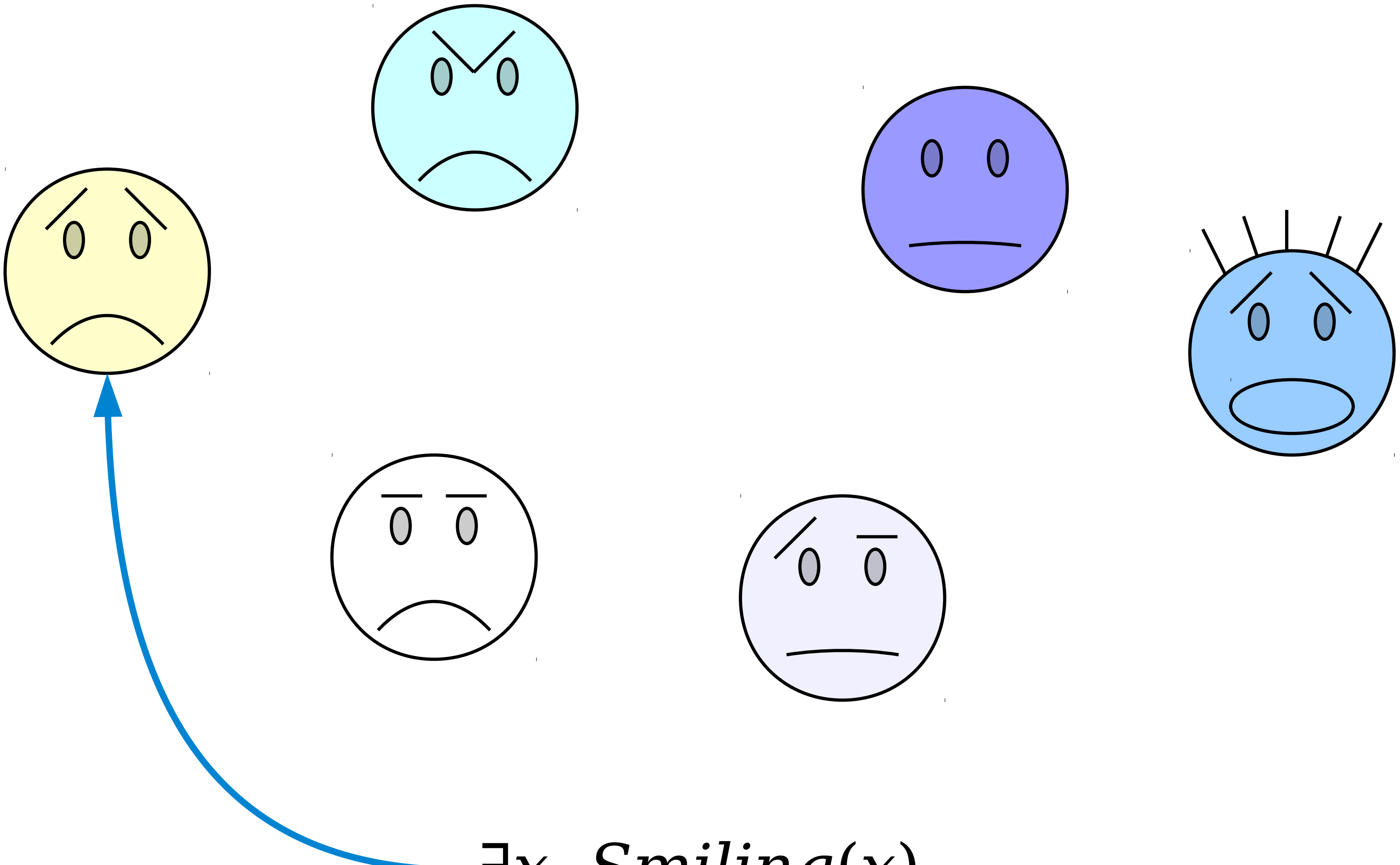

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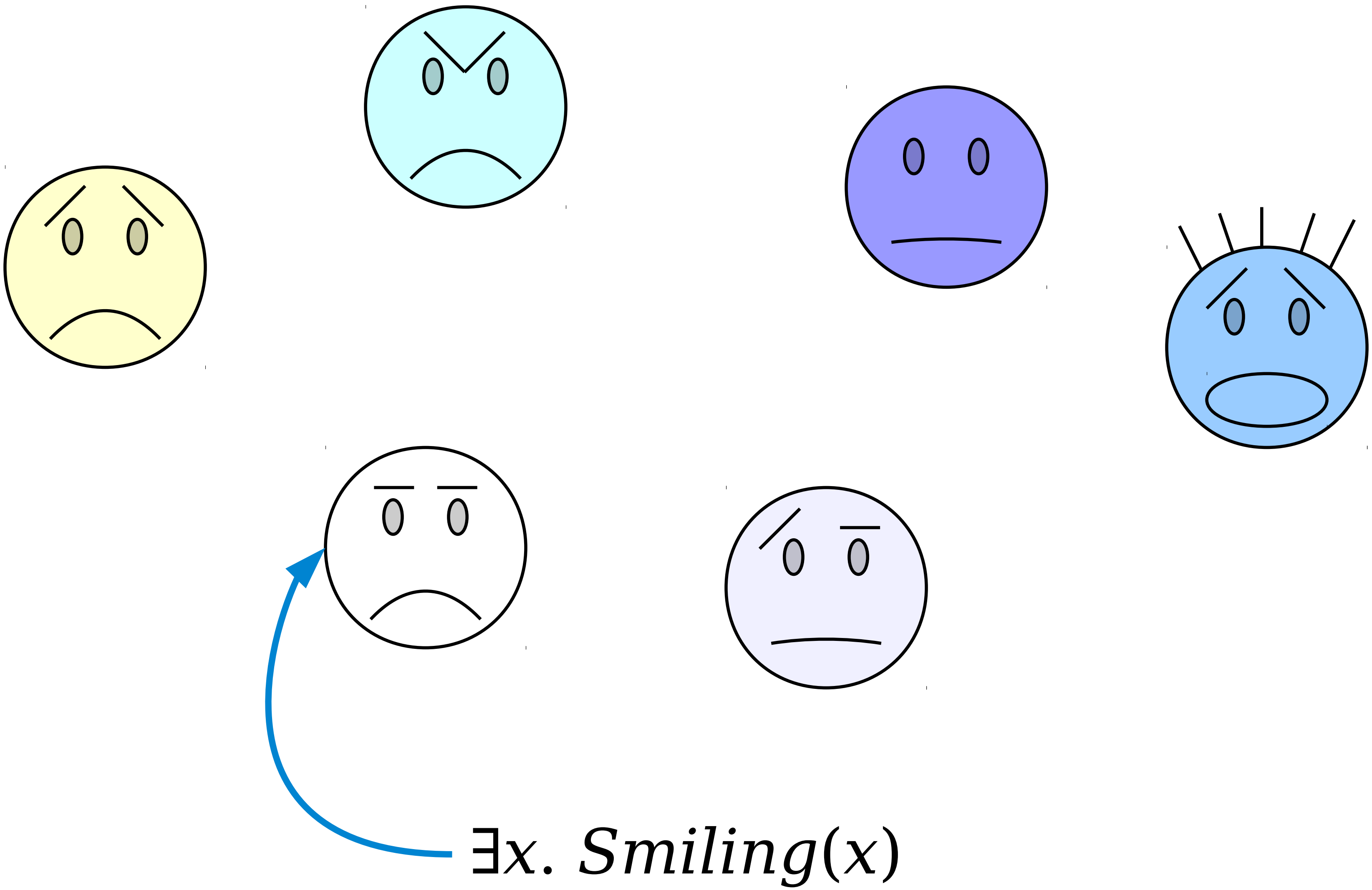

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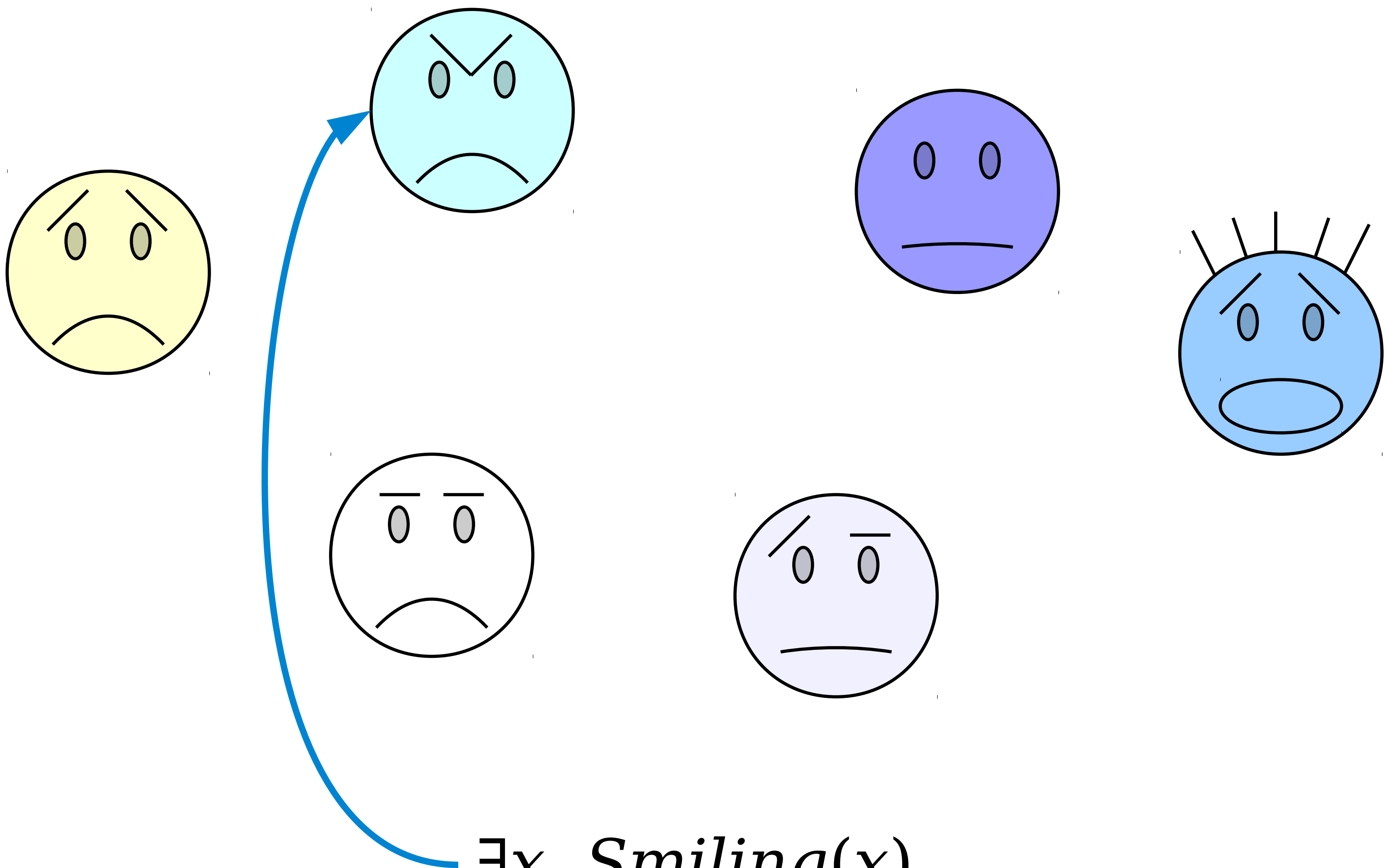

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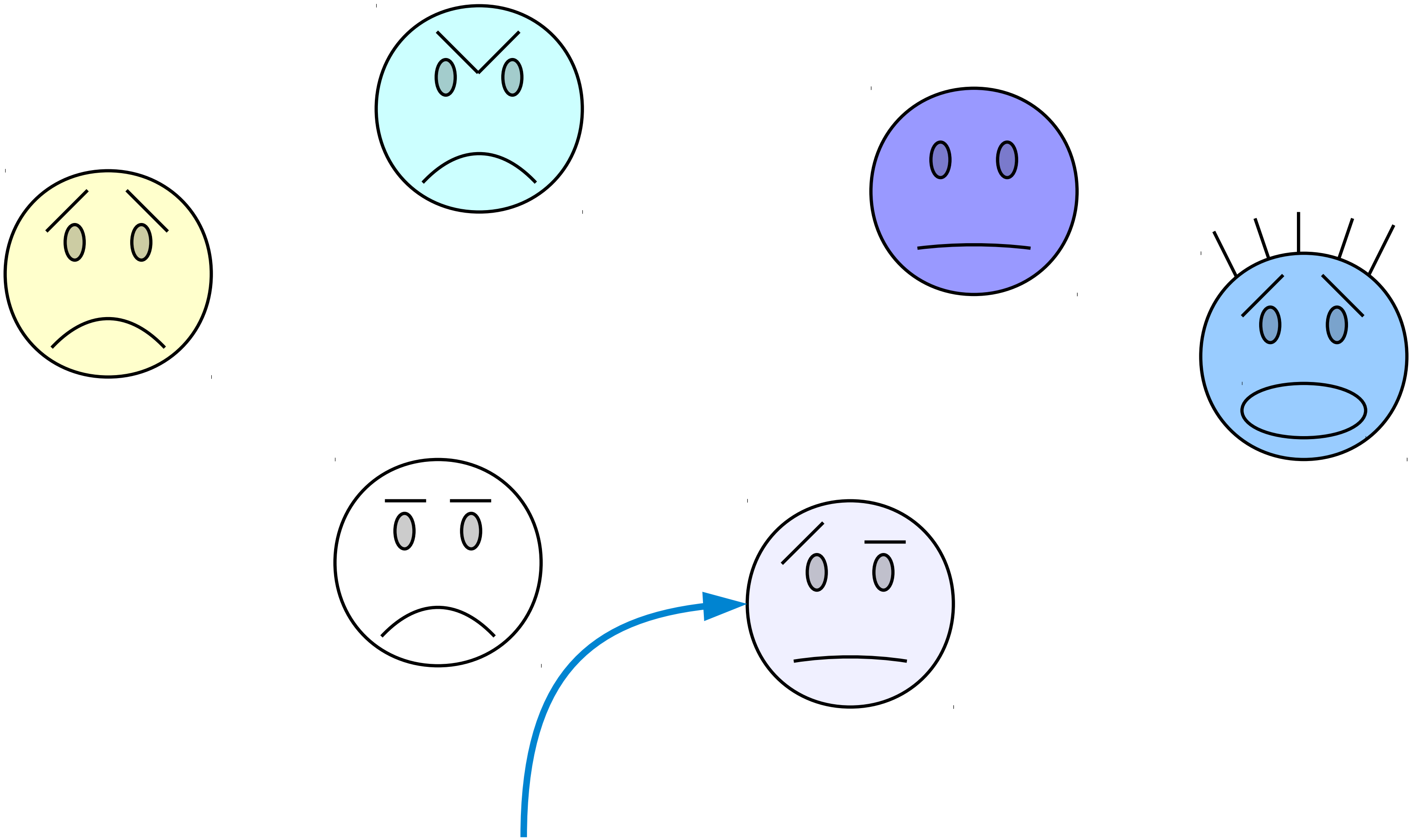

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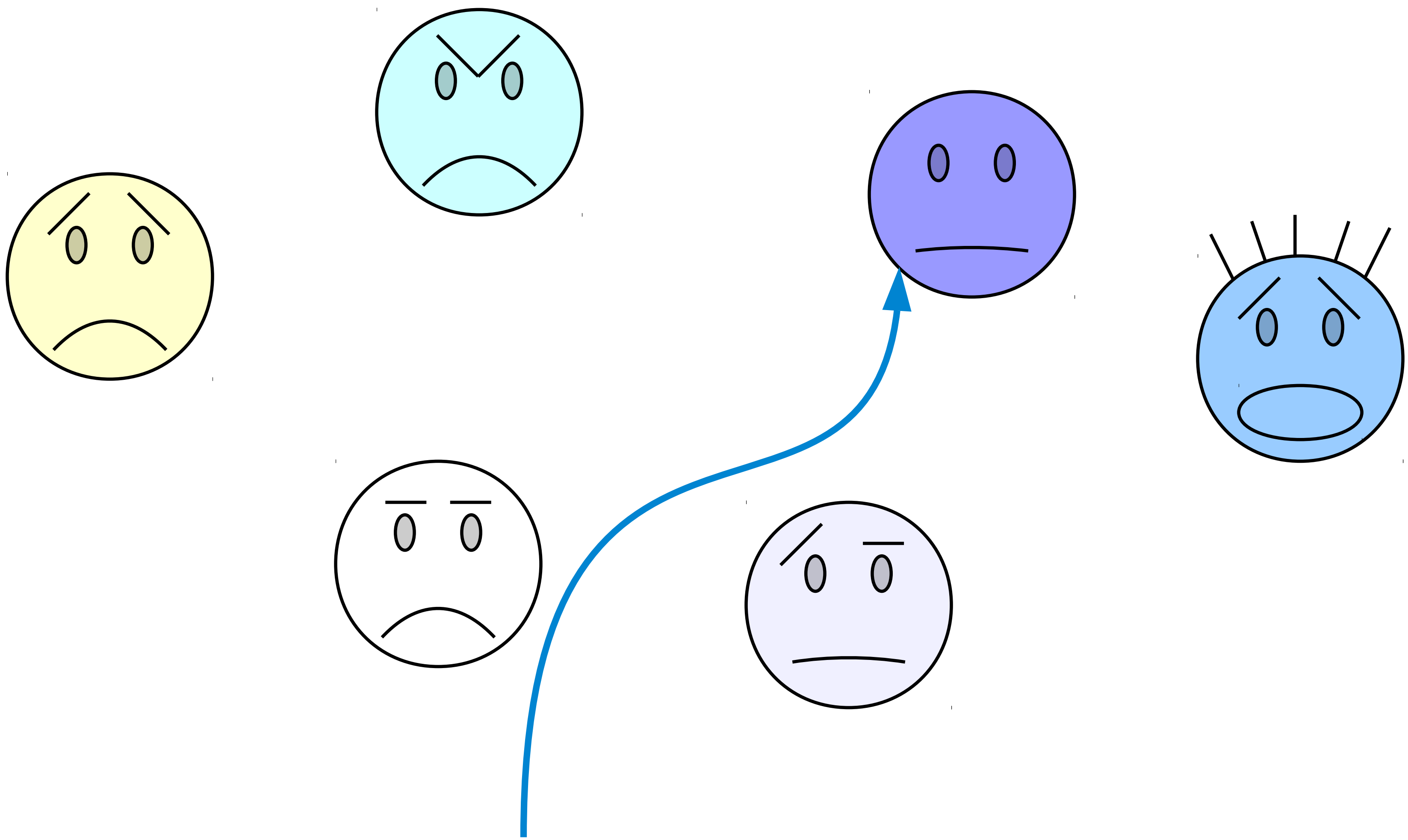

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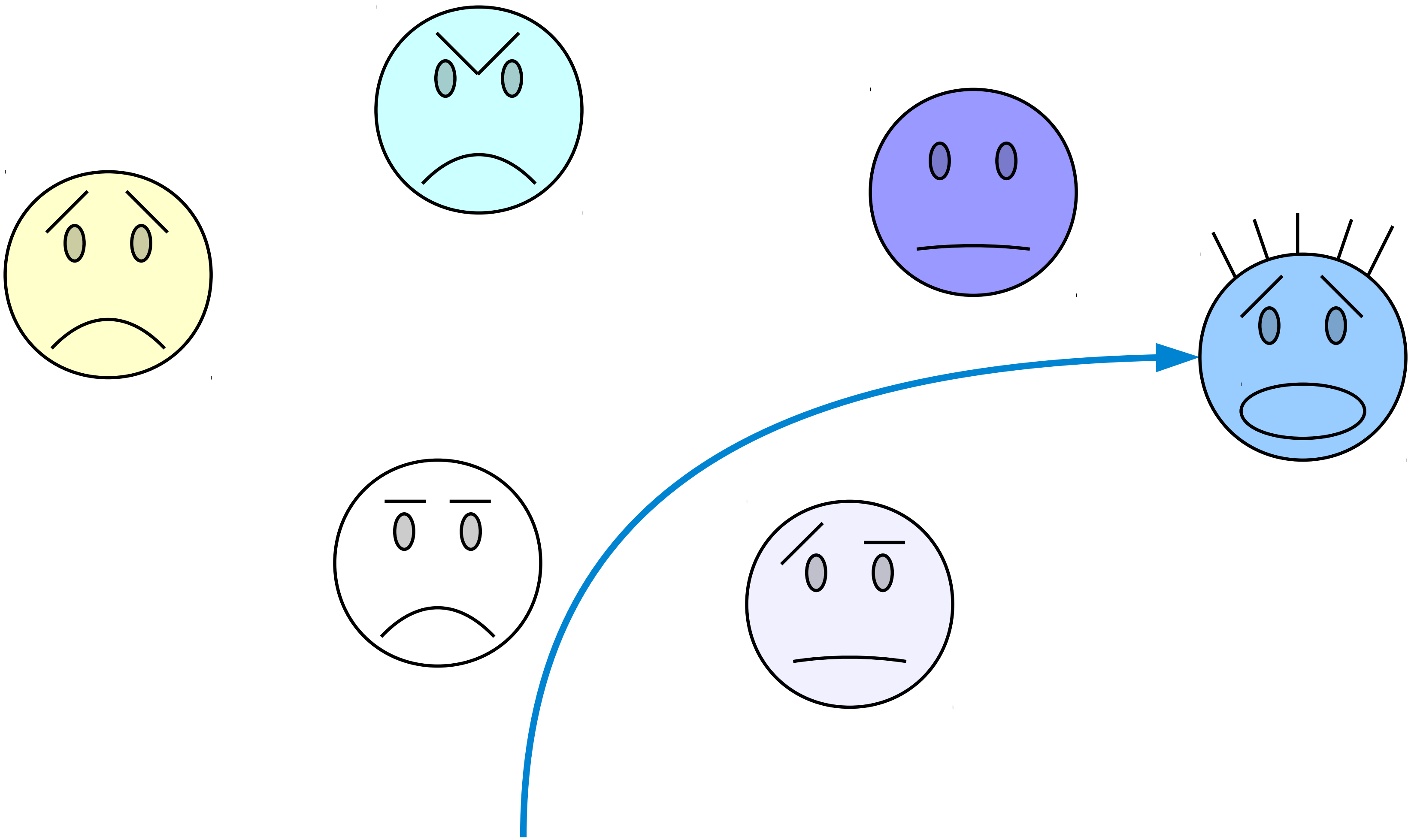
The Existential Quantifier



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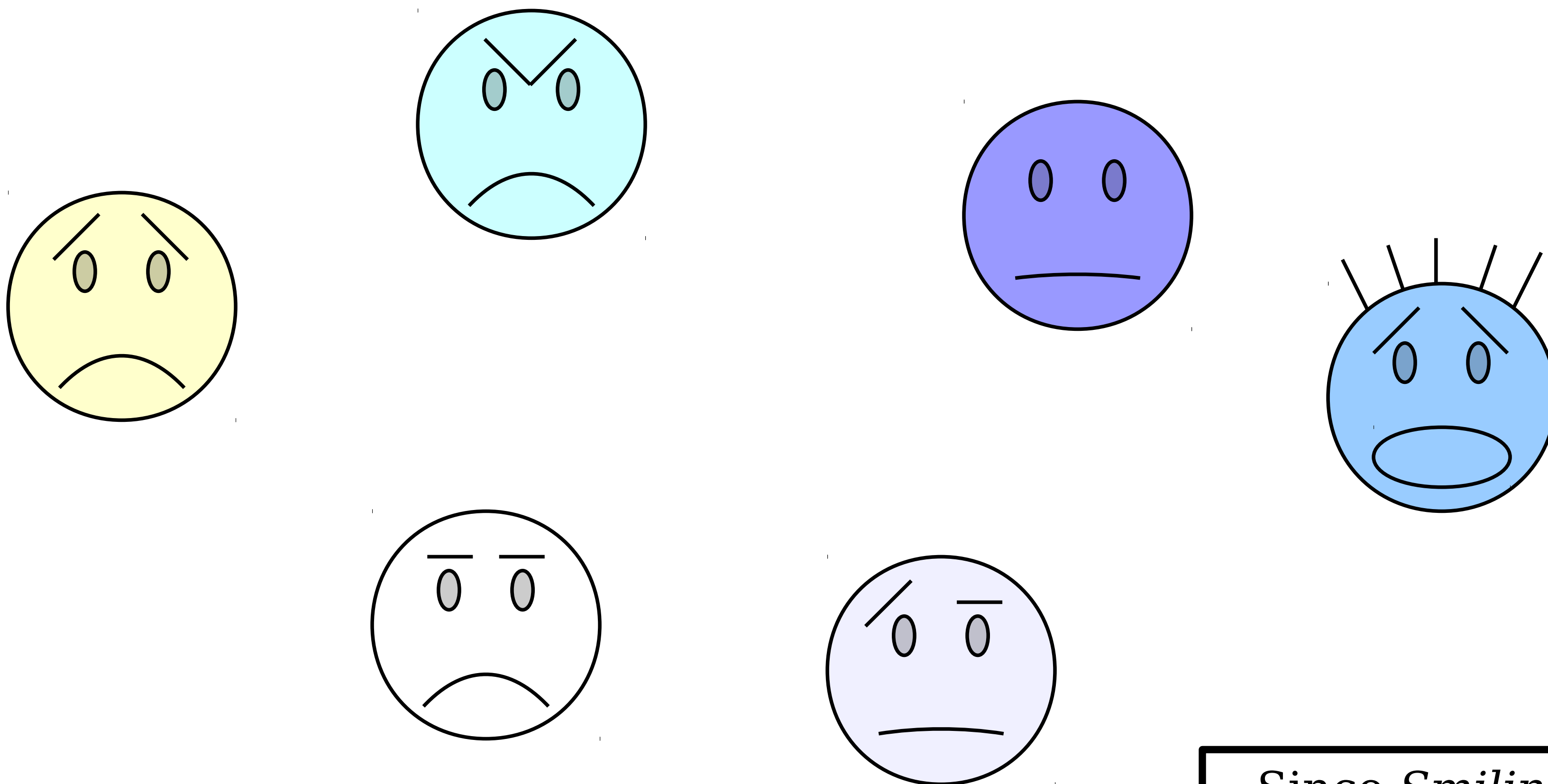
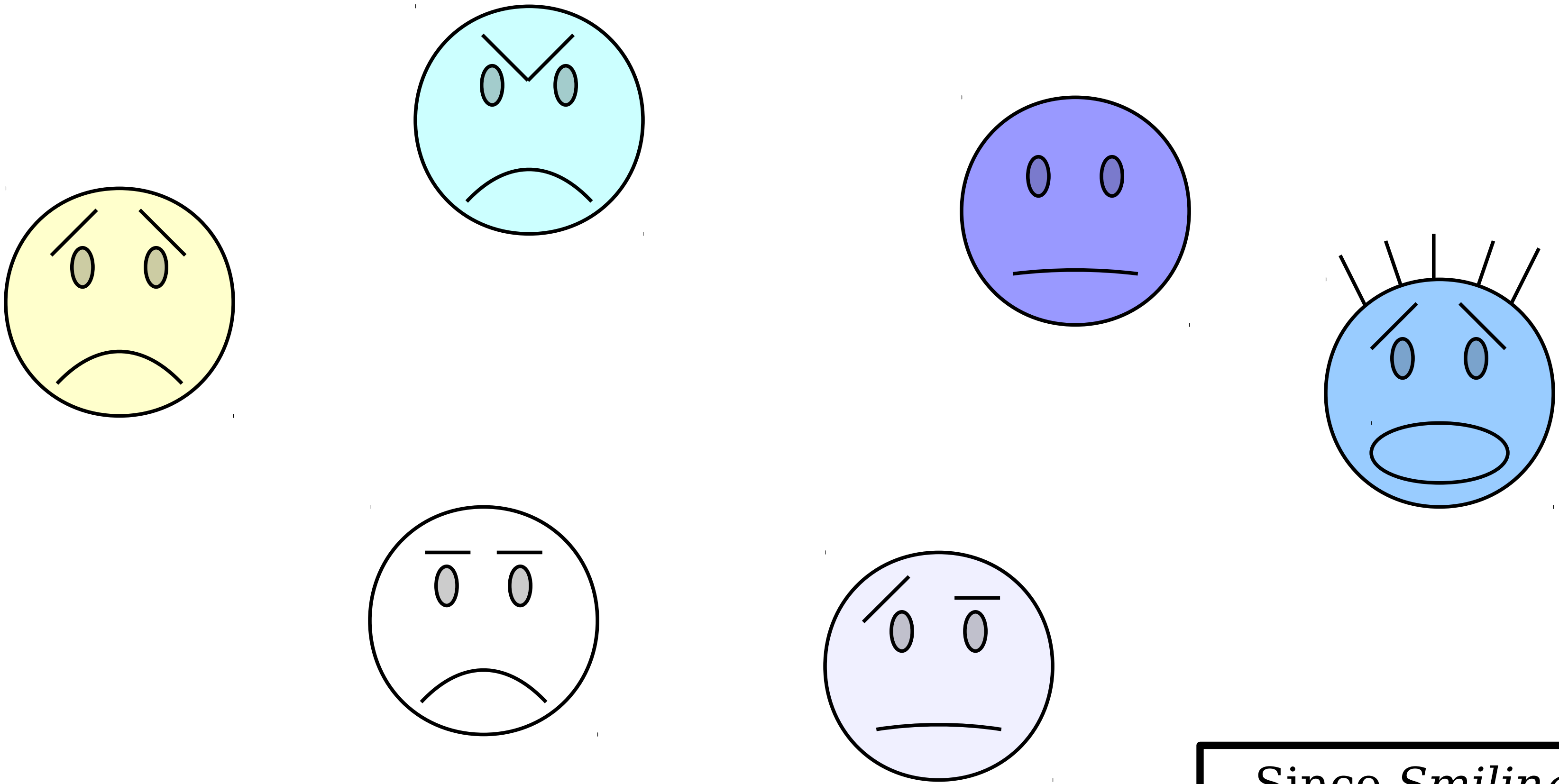
The Existential Quantifier



$\exists x.\ \mathit{Smiling}(x)$

Since $\mathit{Smiling}(x)$ is not true for any choice of x, this statement evaluates to false.

The Existential Quantifier



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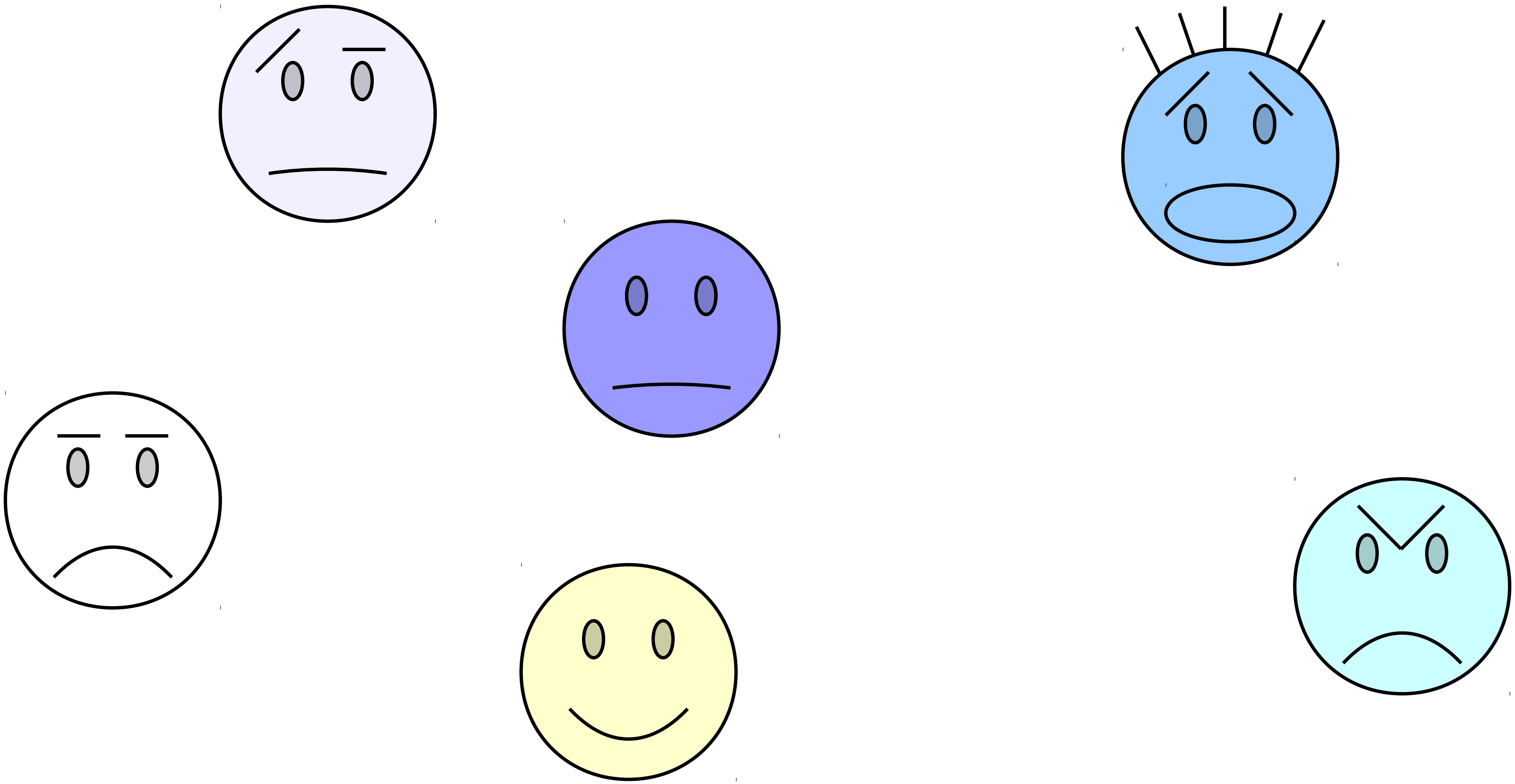
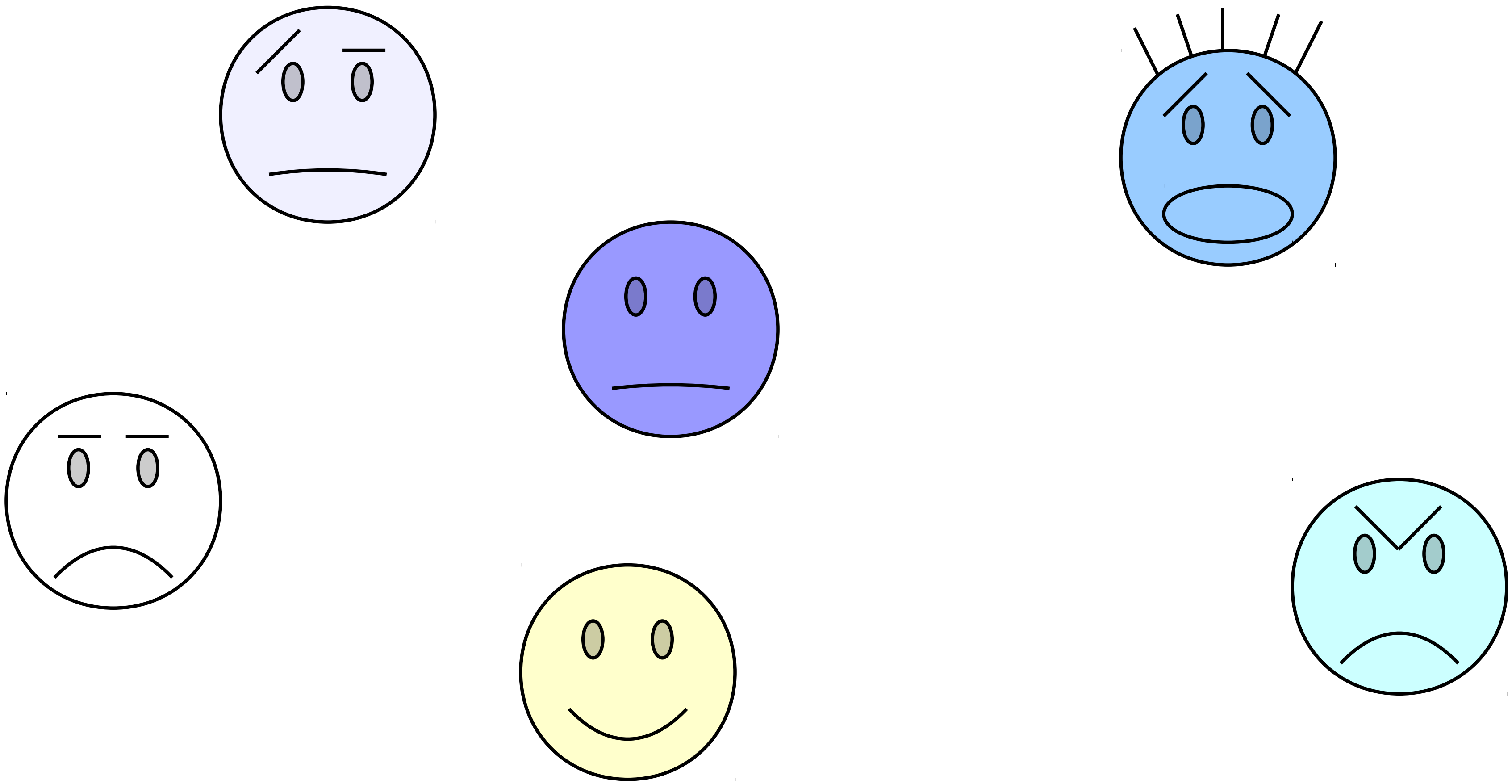


$$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$$

The Existential Quantifier



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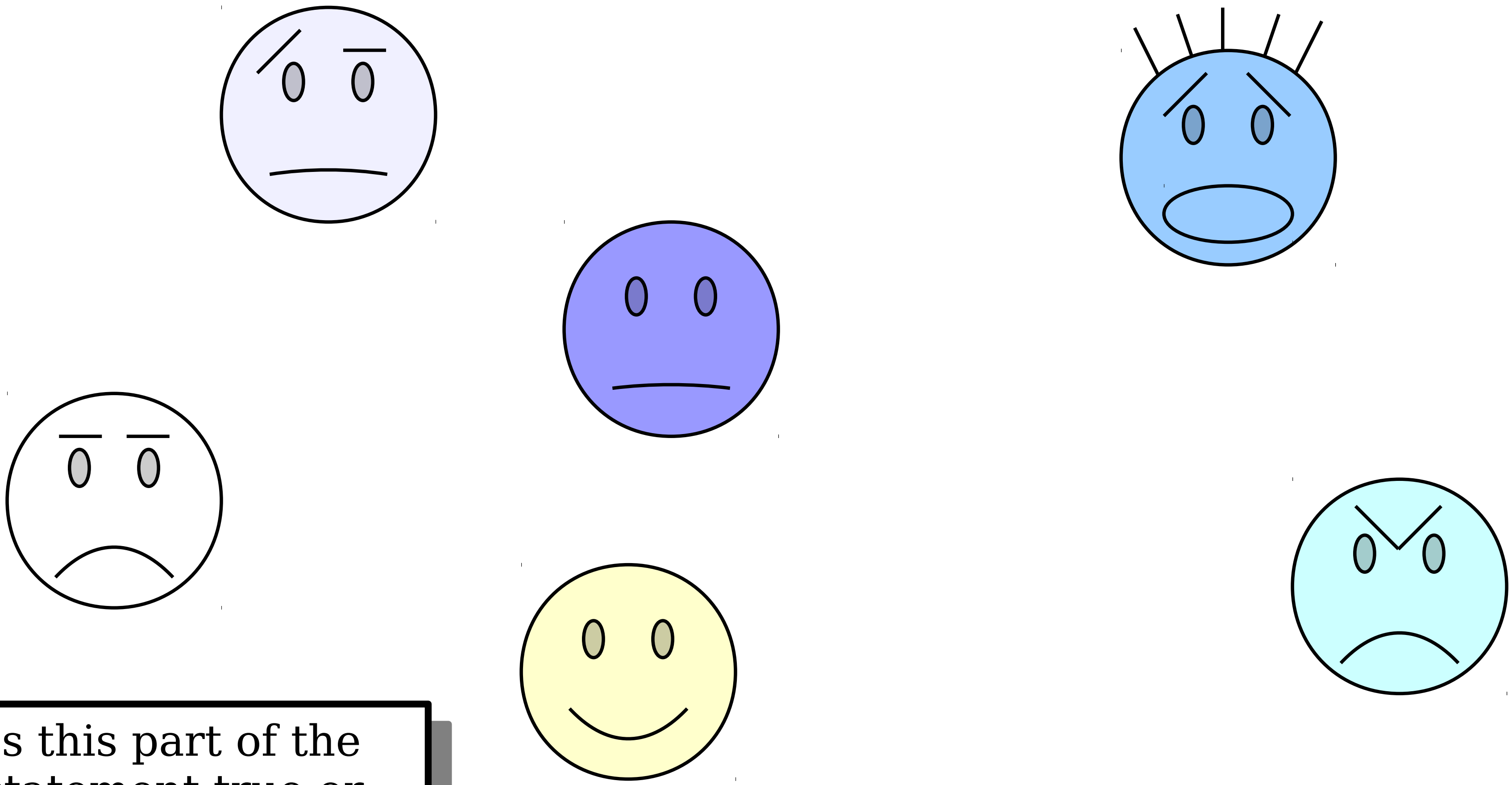
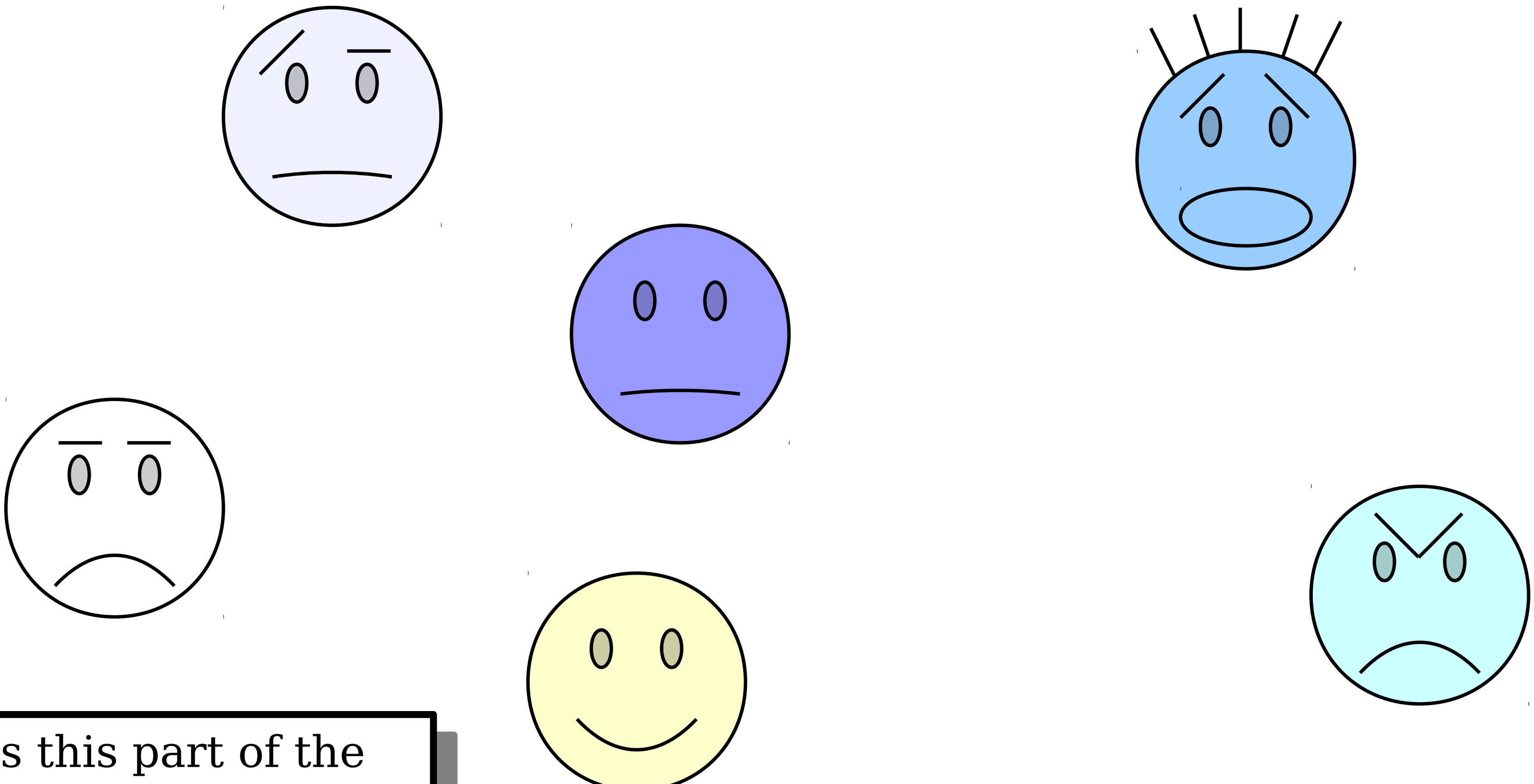
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$

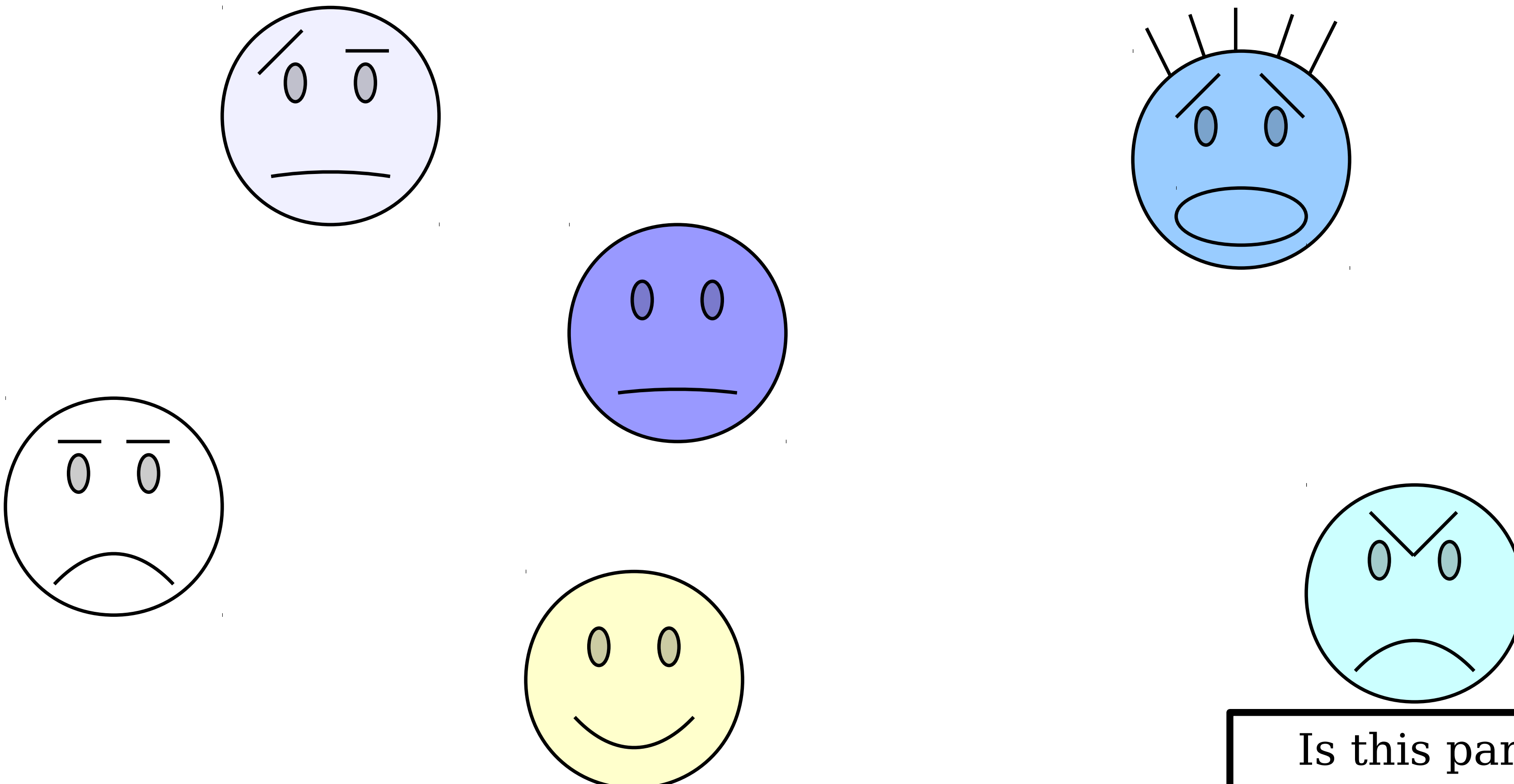
The Existential Quantifier



Is this part of the statement true or false?

$$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$$

The Existential Quantifier



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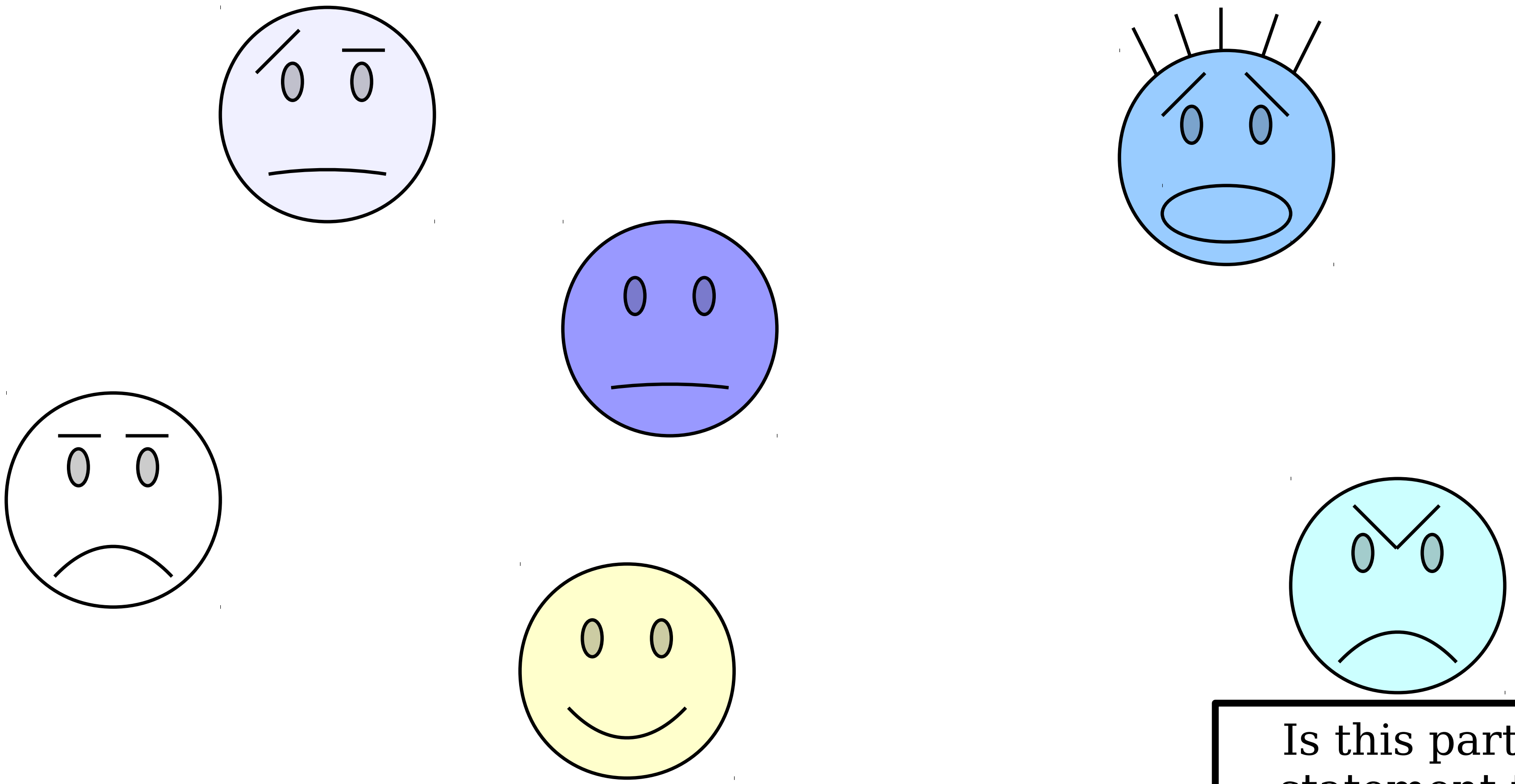
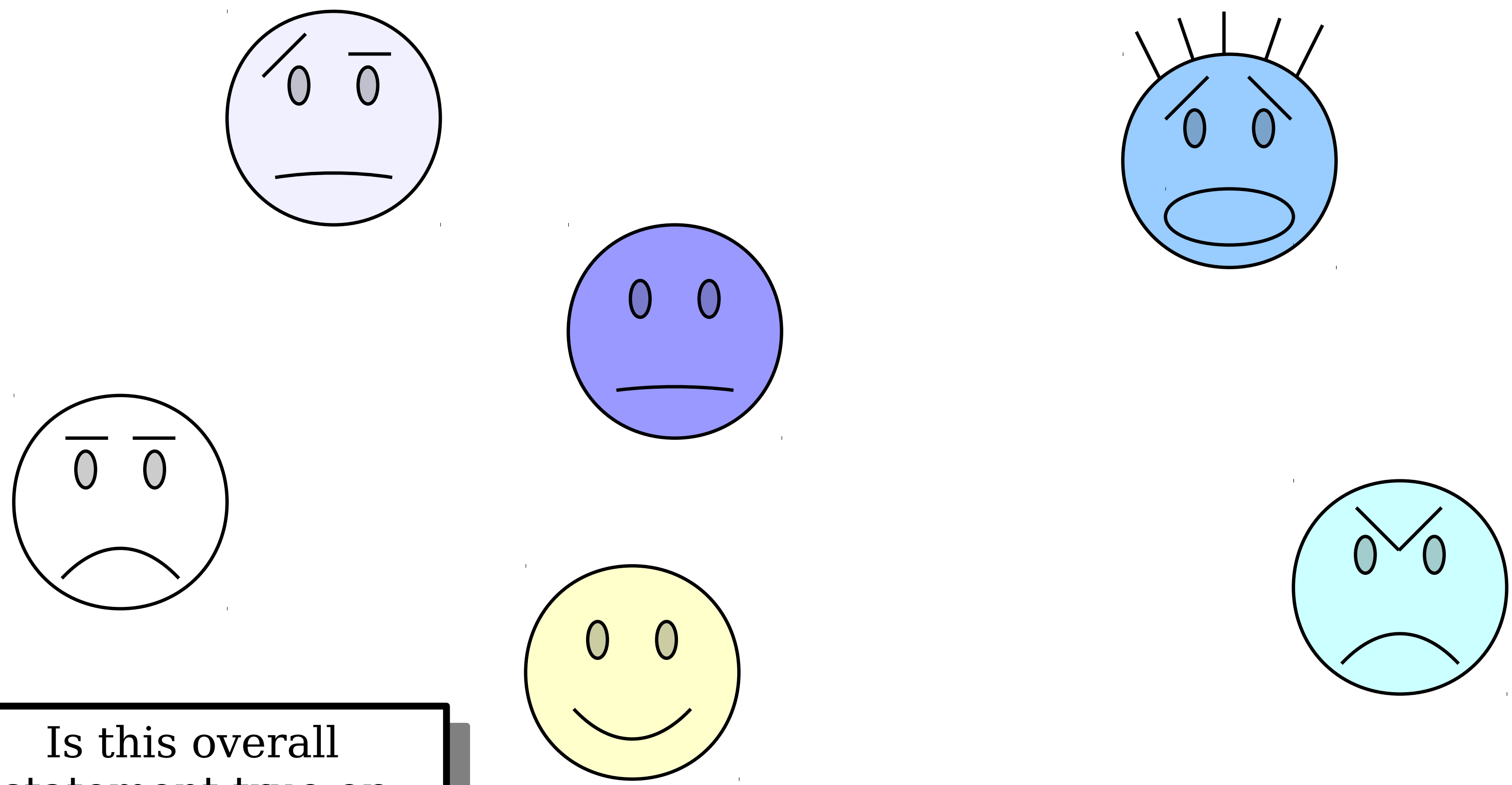


Is this part of the statement true or false?

$(\exists x.\ Smiling(x)) \rightarrow$ ~~$(\exists y.\ WearingHat(y))$~~

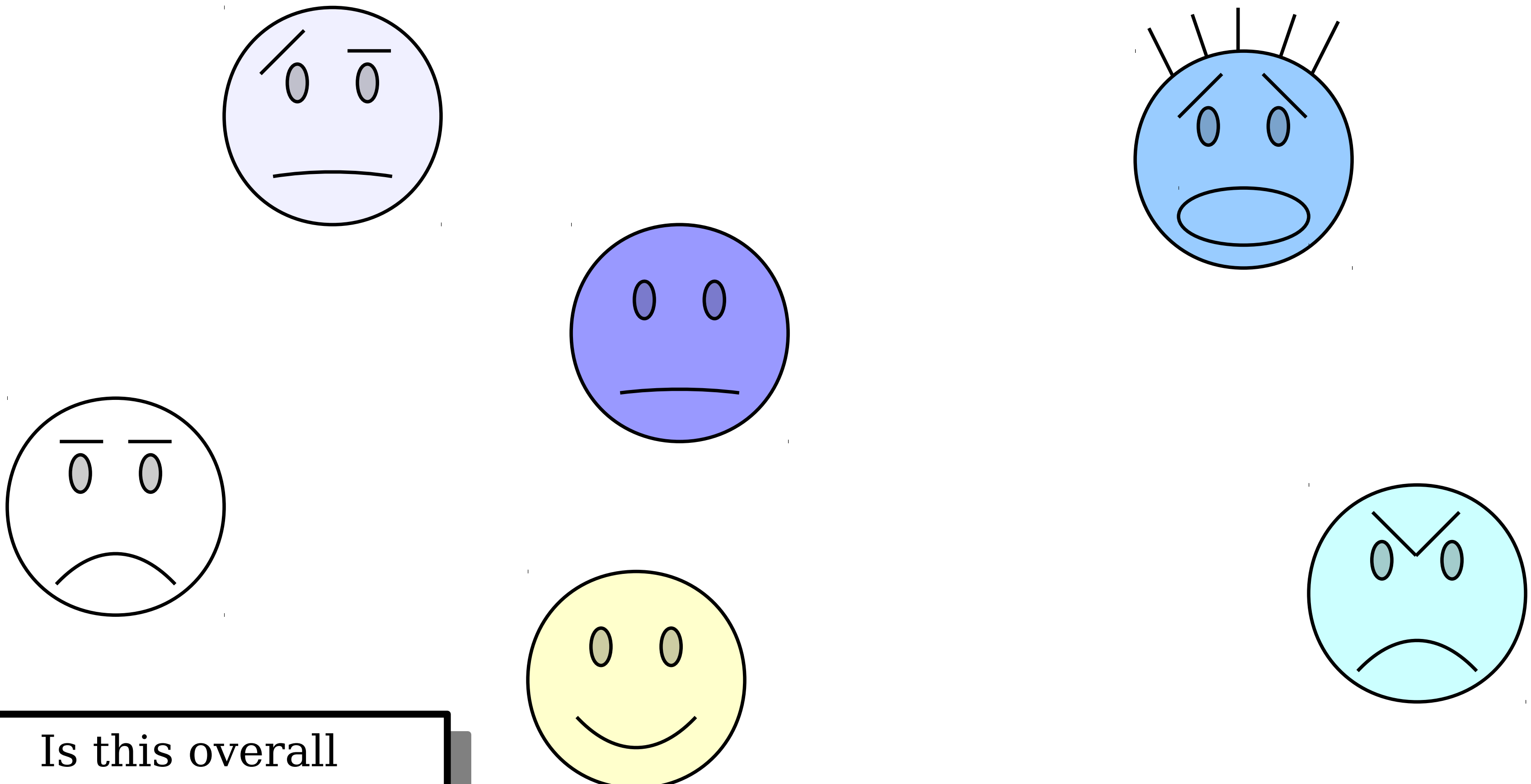
The Existential Quantifier



Is this overall statement true or false?

$(\exists x.\ \mathit{Smiling}(x)) \to$ ~~$(\exists y.\ \mathit{WearingHat}(y))$~~

The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x.\ Smiling(x)) \to (\exists y.\ WearingHat(y))$~~

Fun with Edge Cases

$$\exists x.\ Smiling(x)$$

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

~~$\exists x.\ Smiling(x)$~~

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n. \; (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$$

“For any natural number n,
n is even if and only if n^2 is even”

$$\forall n.\ (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier***
and says “for any choice of n,
the following is true.”

The Universal Quantifier

- A statement of the form

$$\forall x.\ \textit{some-formula}$$

  is true if, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.

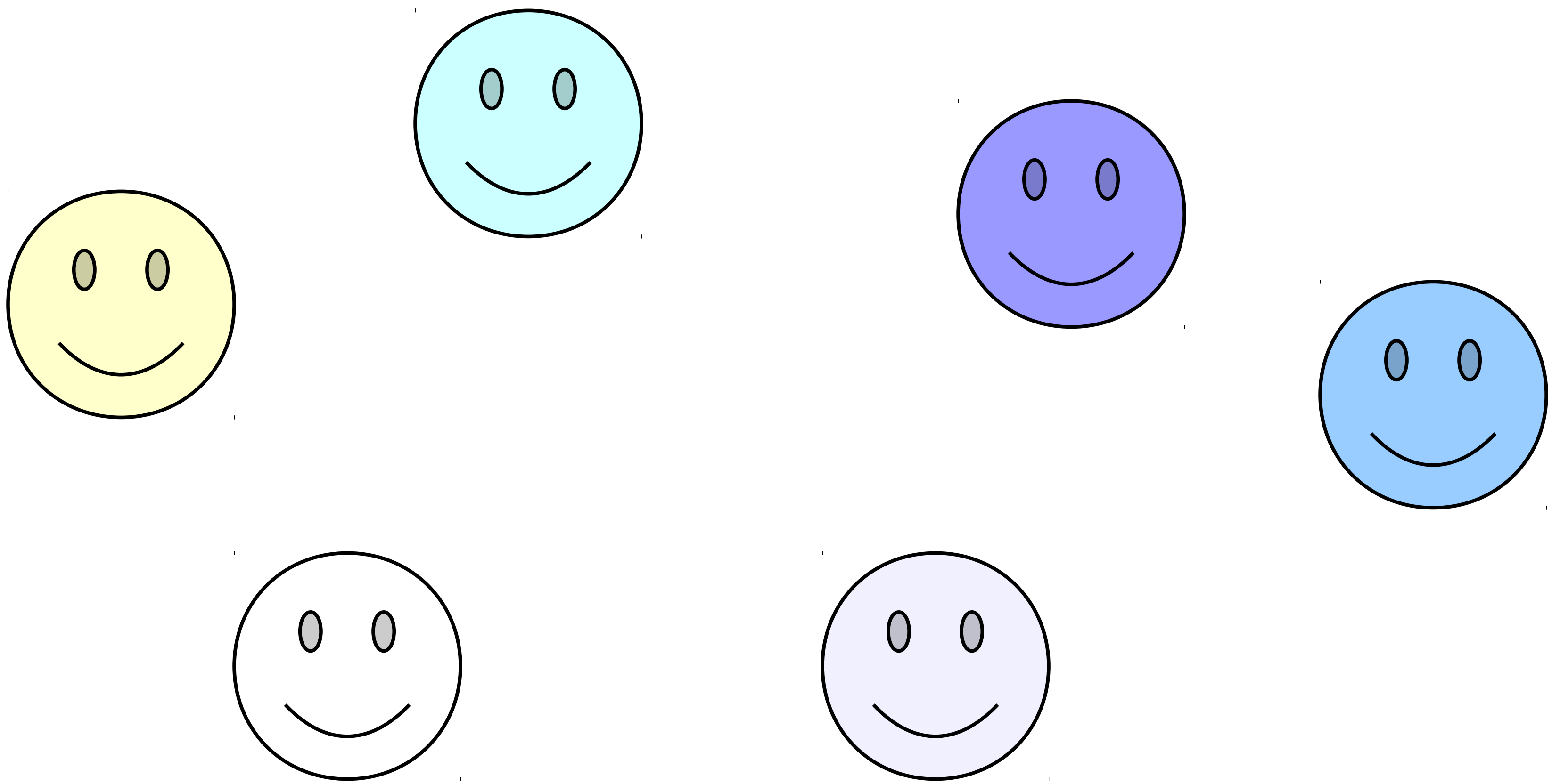
- Examples:

$$\forall p.\ (Puppy(p) \to Cute(p))$$

$$\forall a.\ (EatsPlants(a) \lor EatsAnimals(a))$$

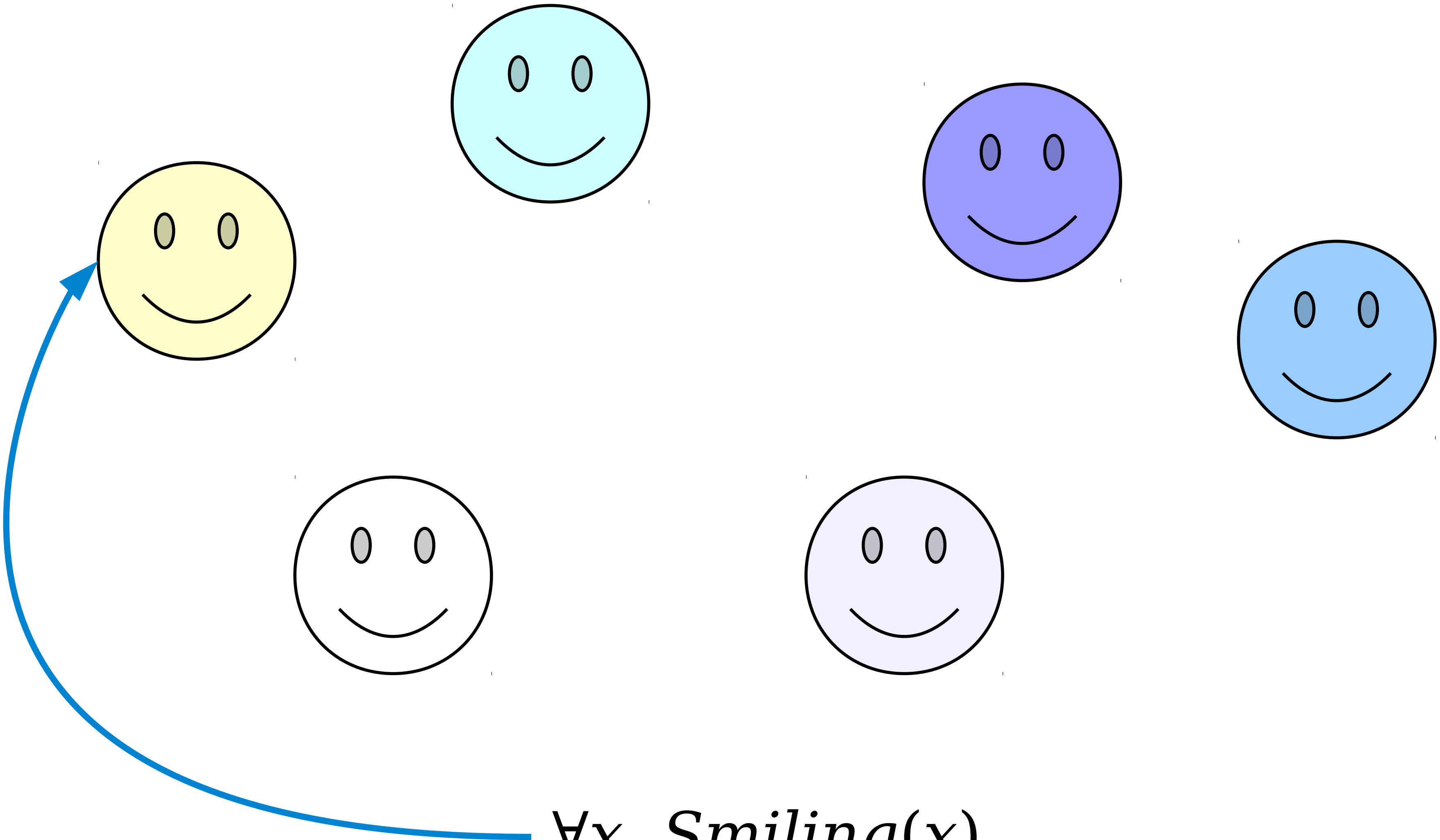
$$Tallest(SultanK\ddot{o}sen) \to \forall x.\ (SultanK\ddot{o}sen \neq x \to ShorterThan(x, SultanK\ddot{o}sen))$$

The Universal Quantifier



$$\forall x.\ Smiling(x)$$

The Universal Quantifier



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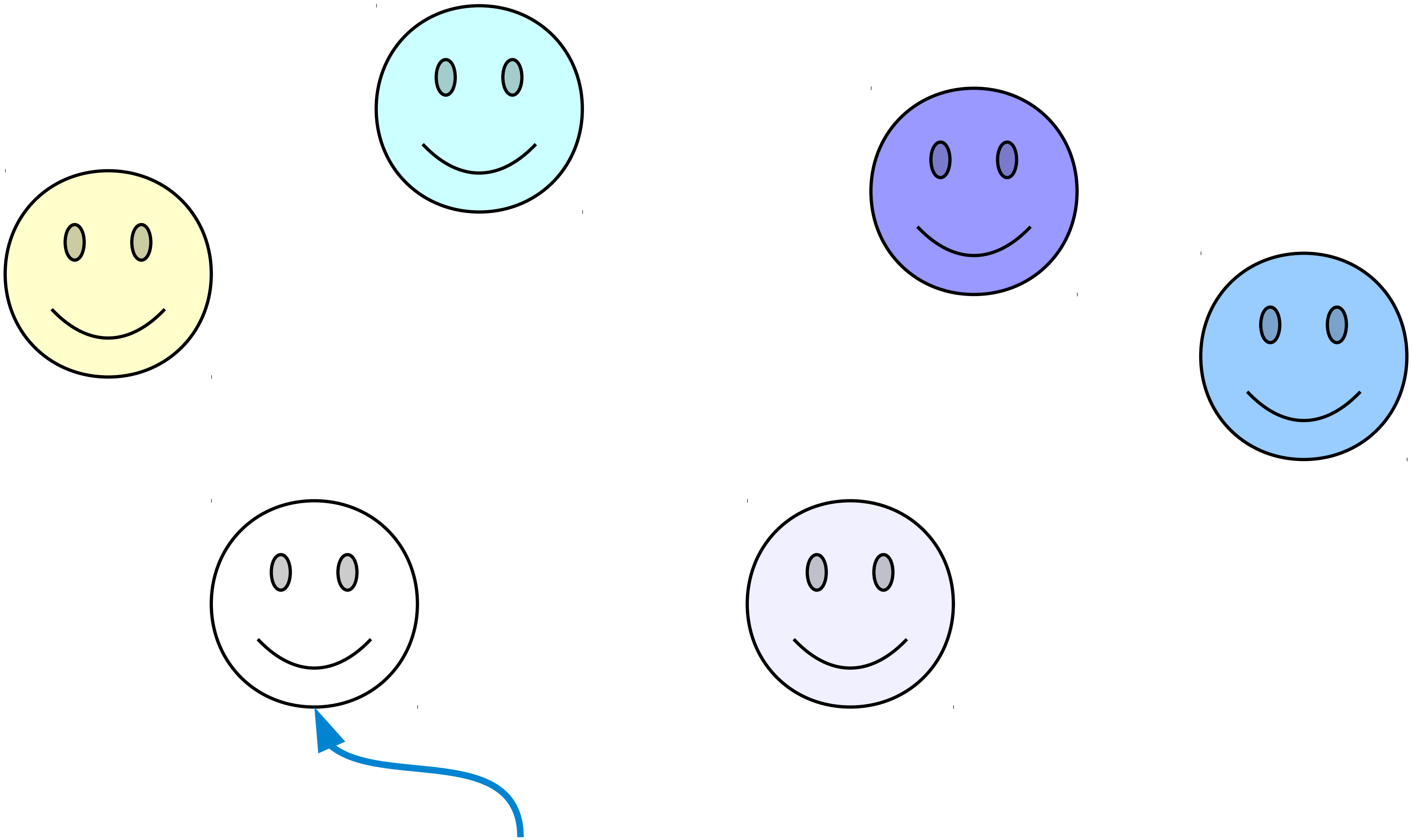

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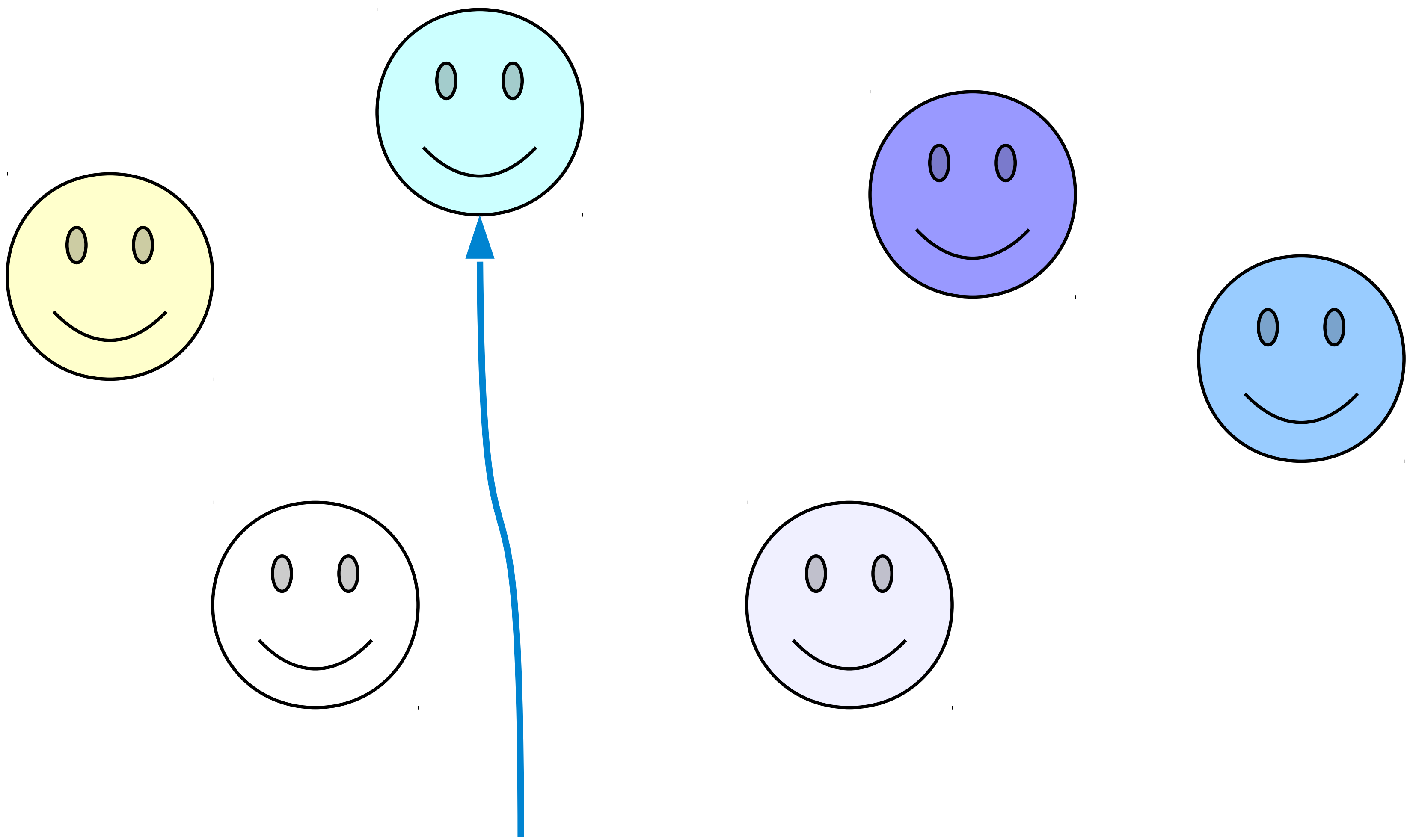

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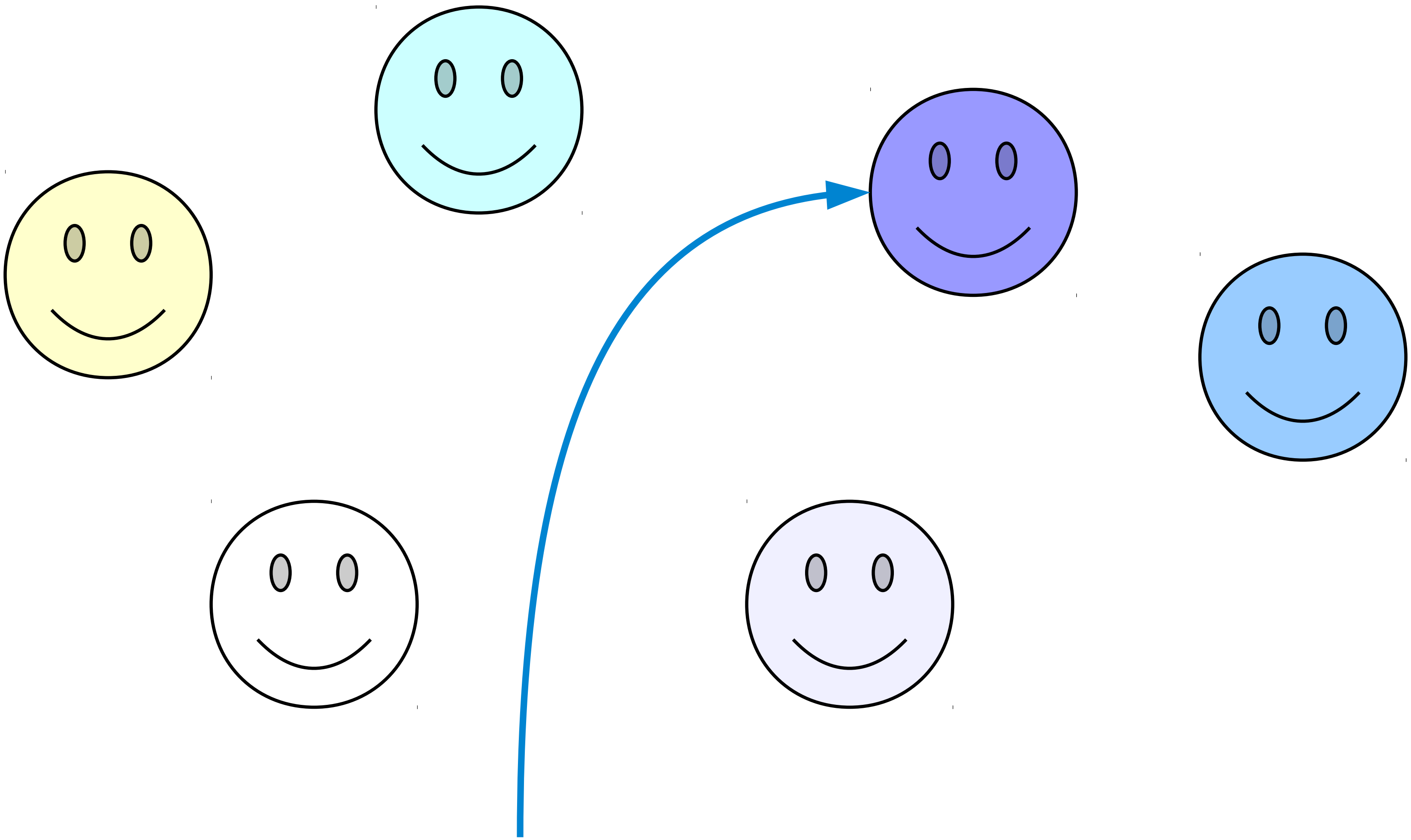

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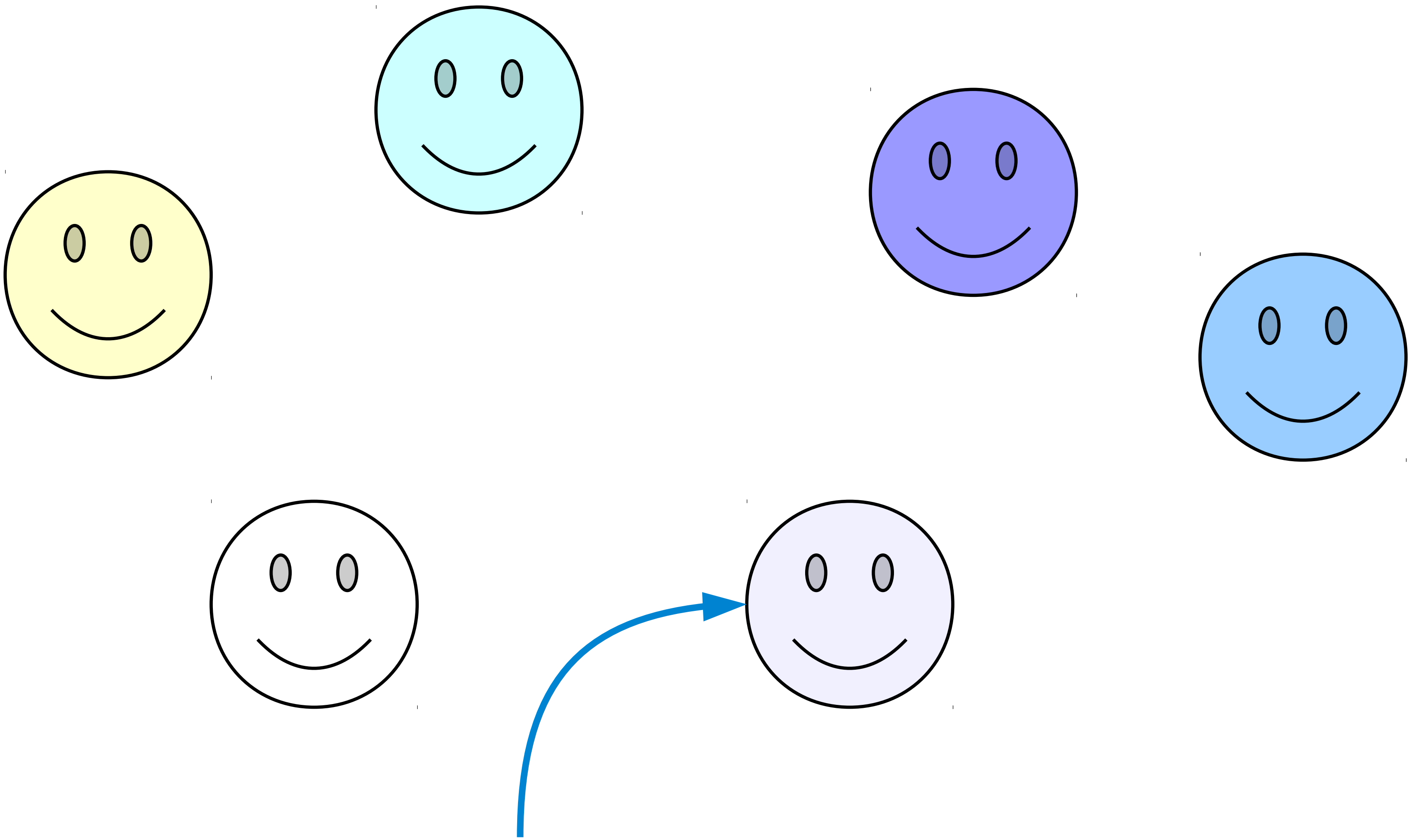

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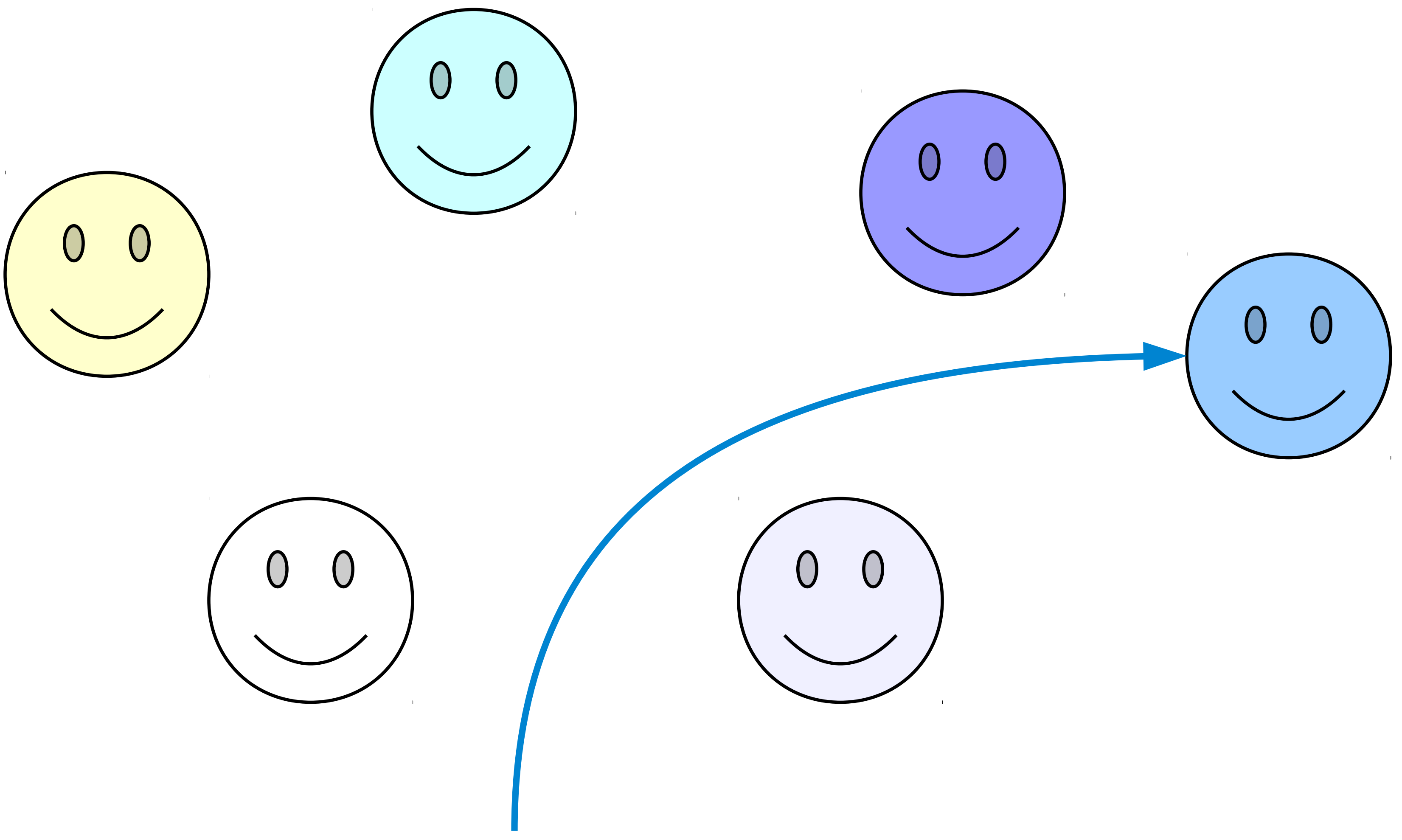
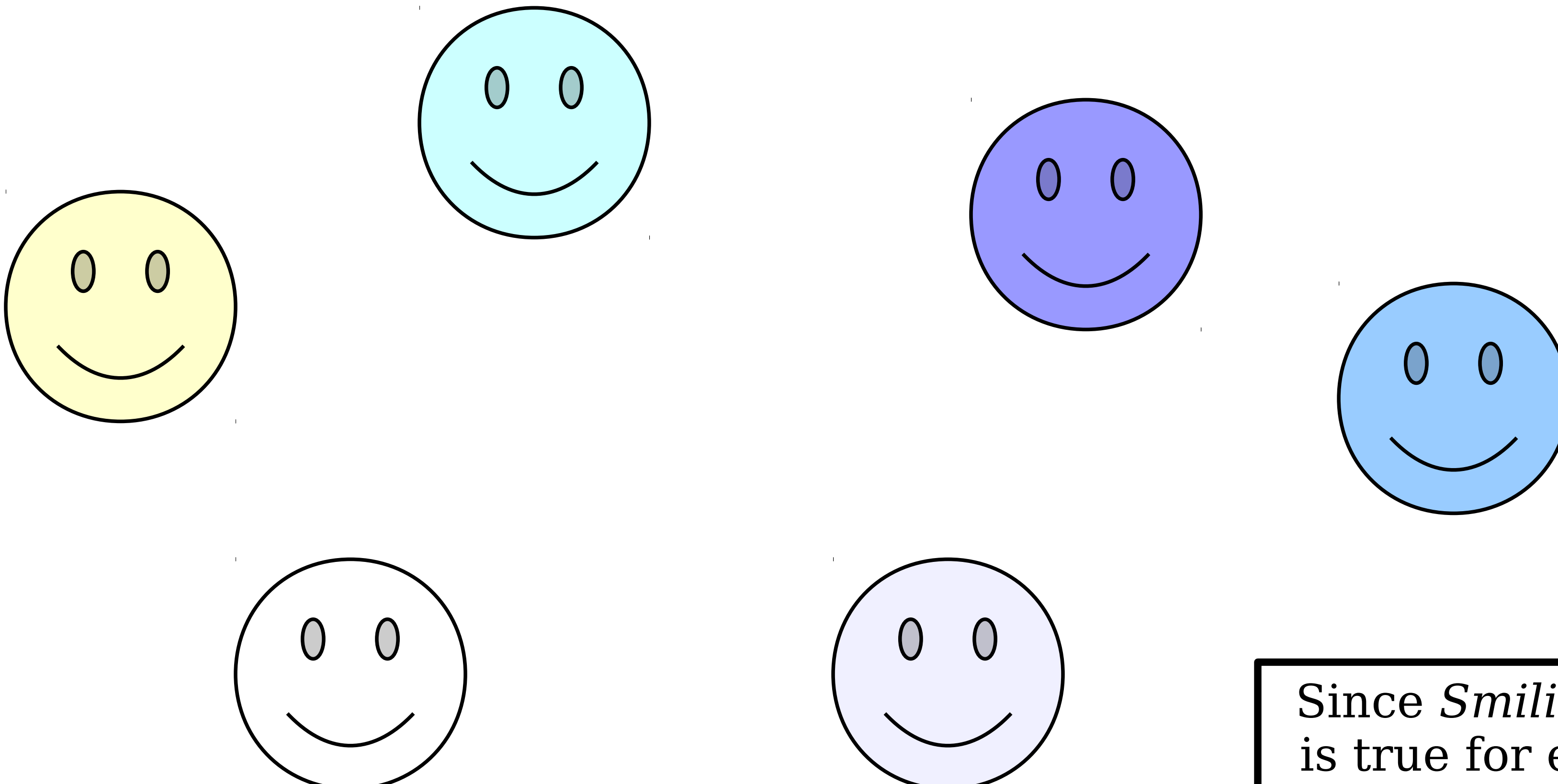
The Universal Quantifier



$\forall x.\ Smiling(x)$

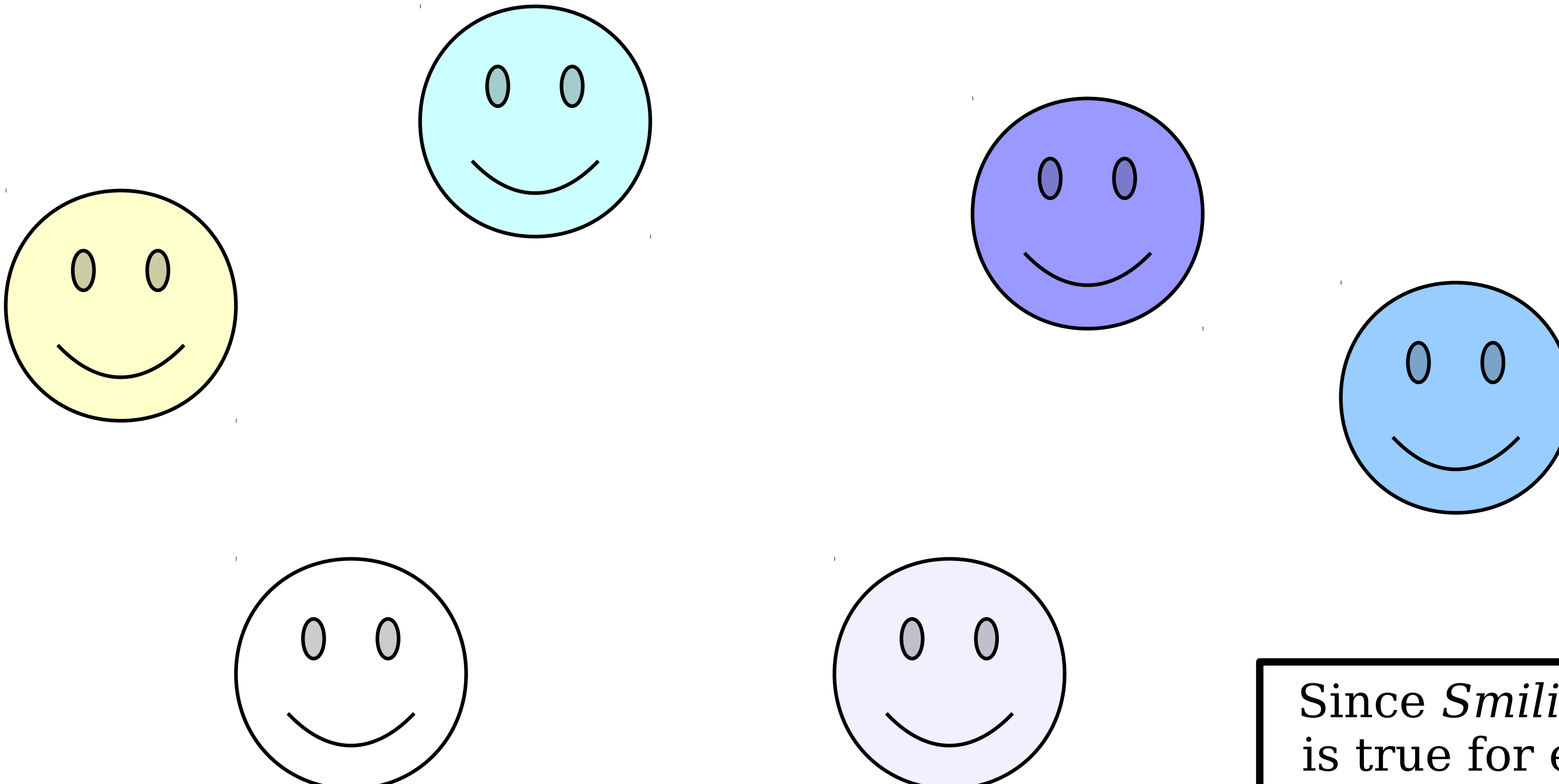
The Universal Quantifier



Since *Smiling*(x) is true for every choice of x, this statement evaluates to true.

$$\forall x.\ Smiling(x)$$

The Universal Quantifier



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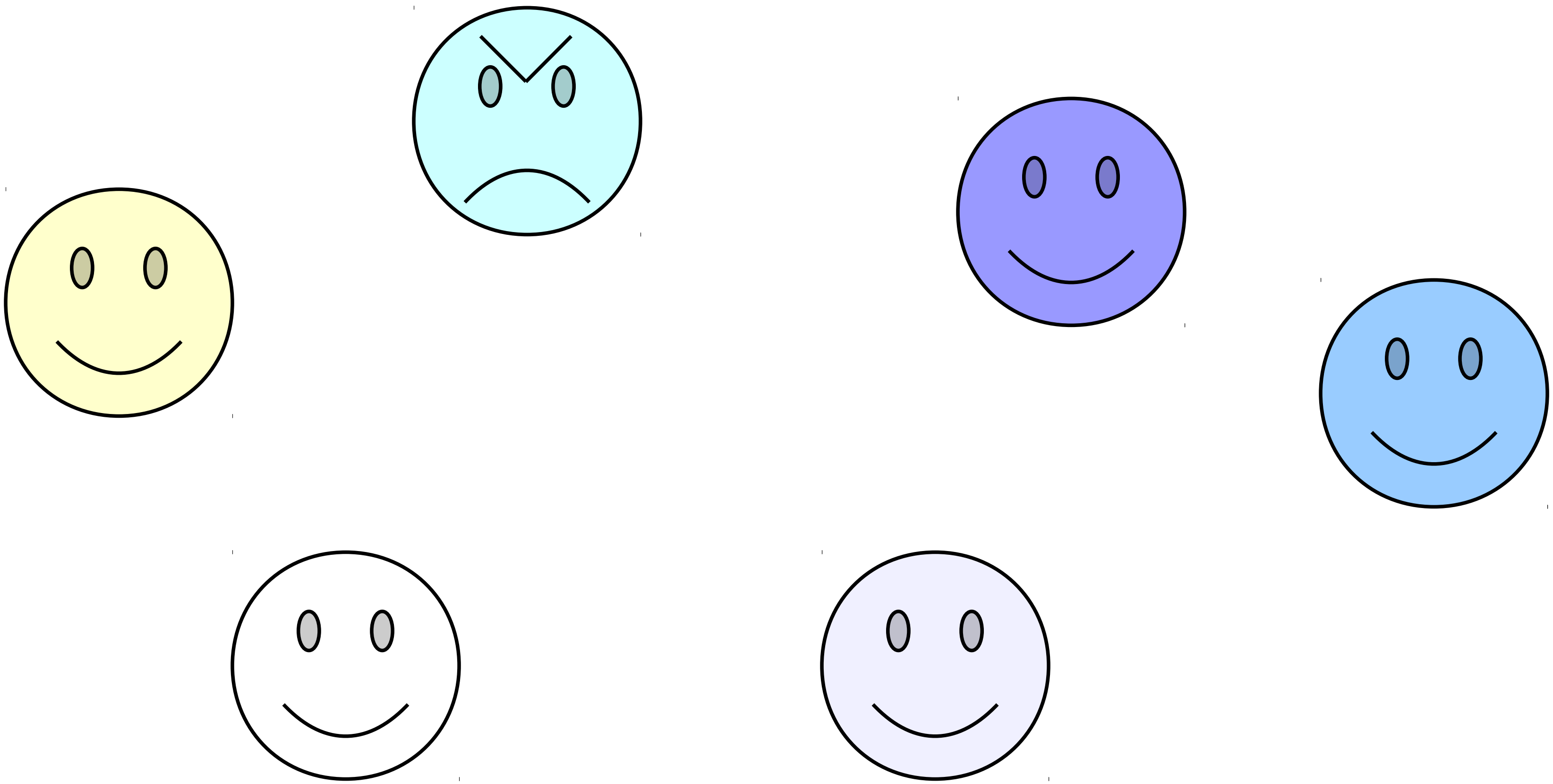

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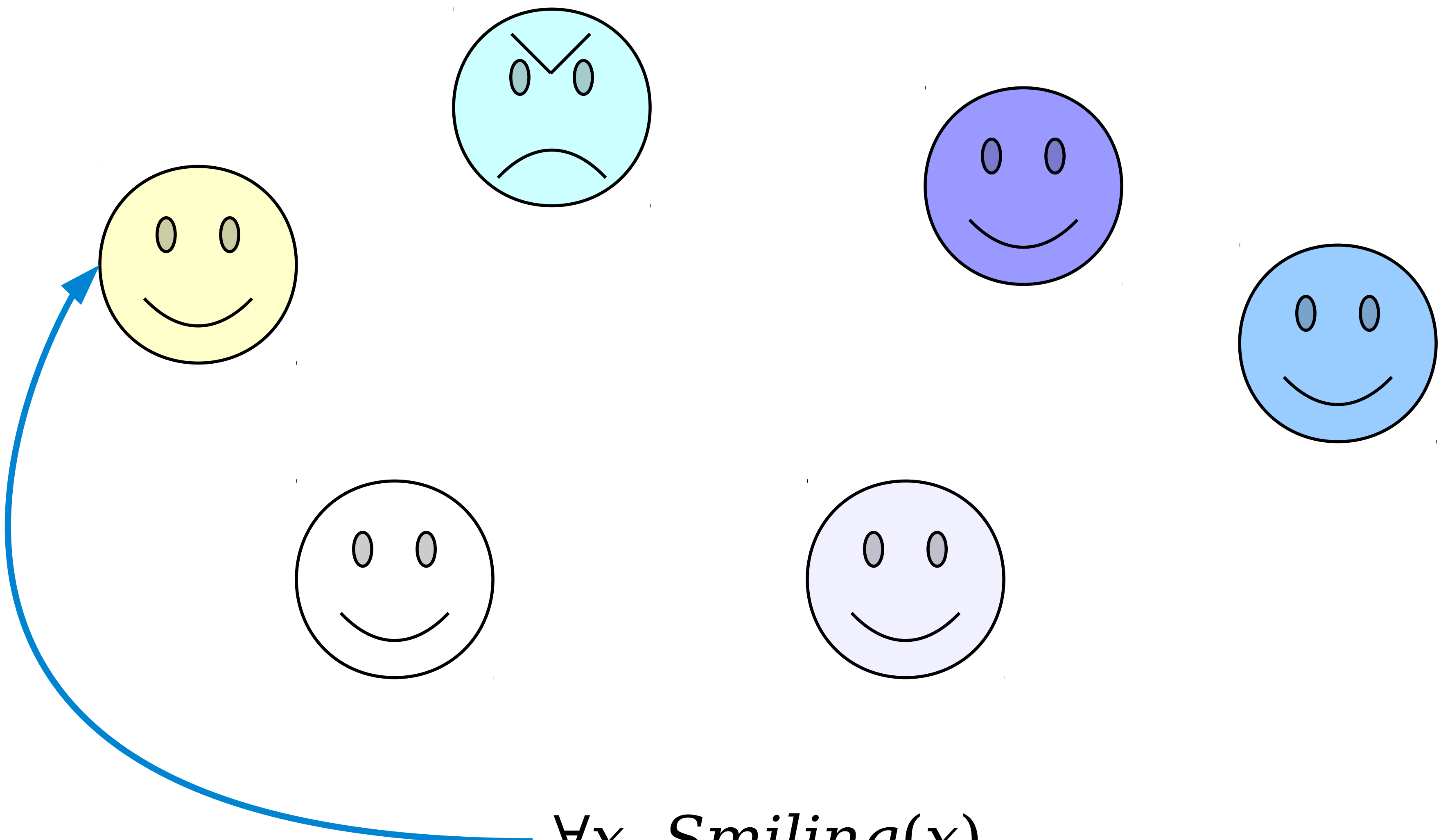
The Universal Quantifier



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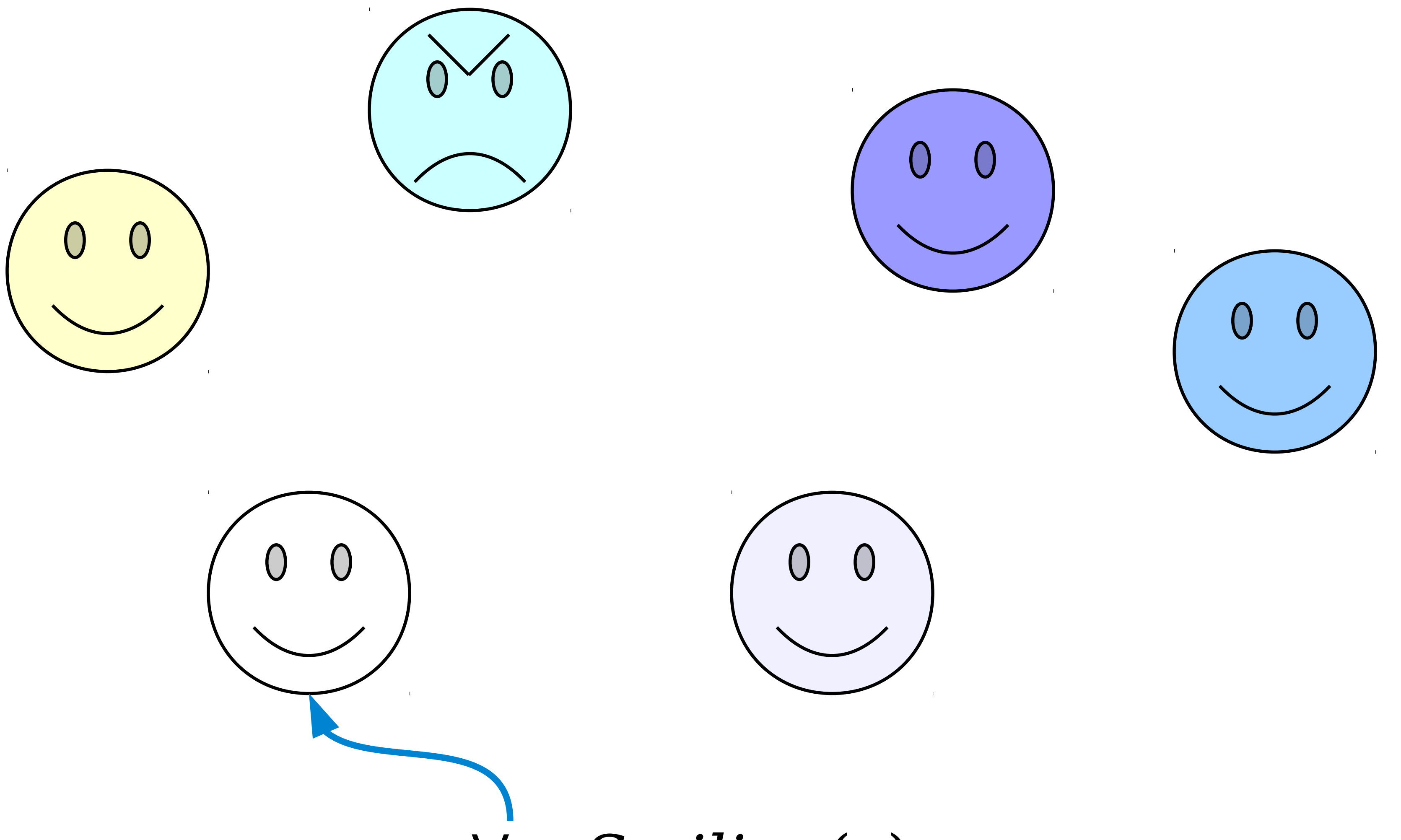

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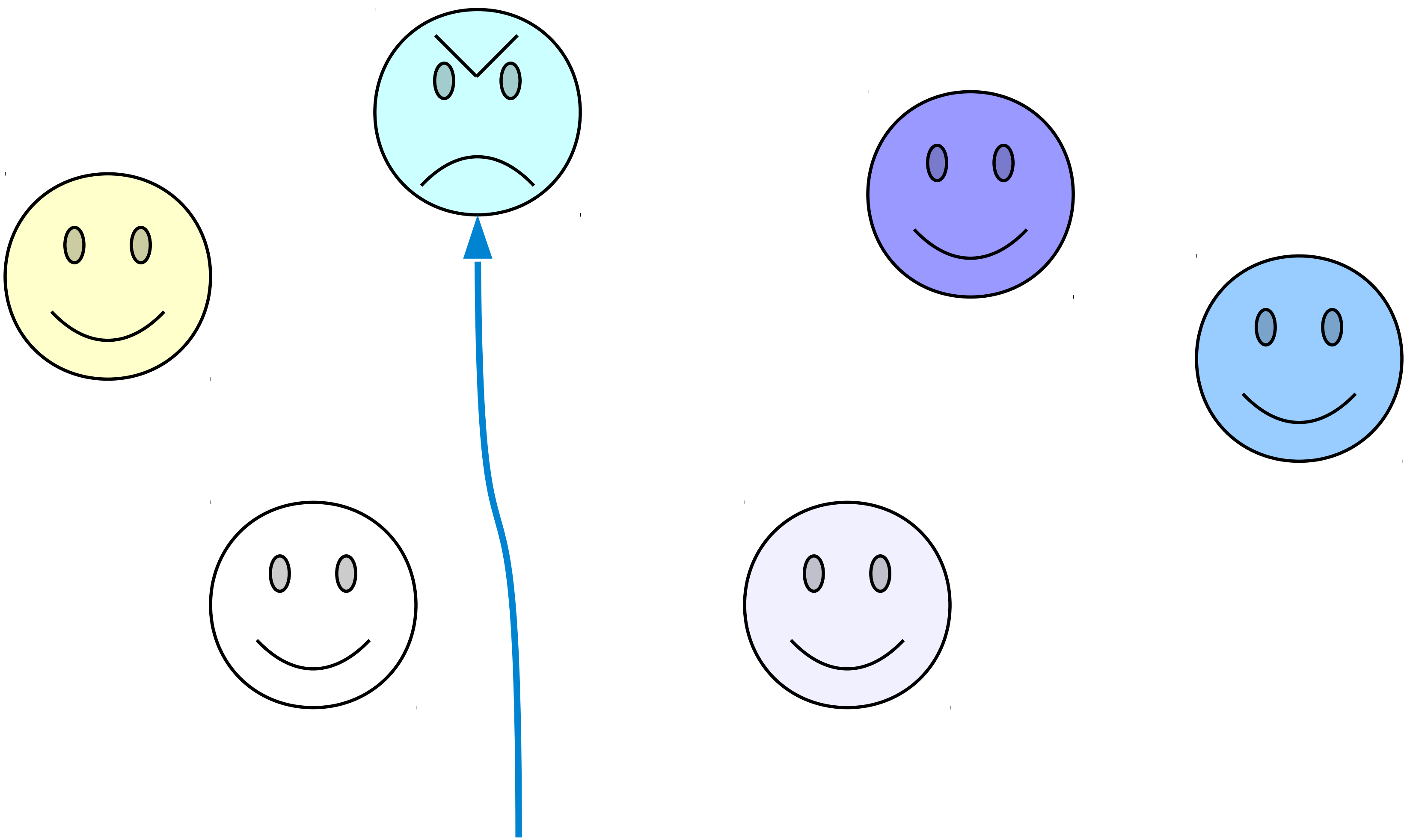

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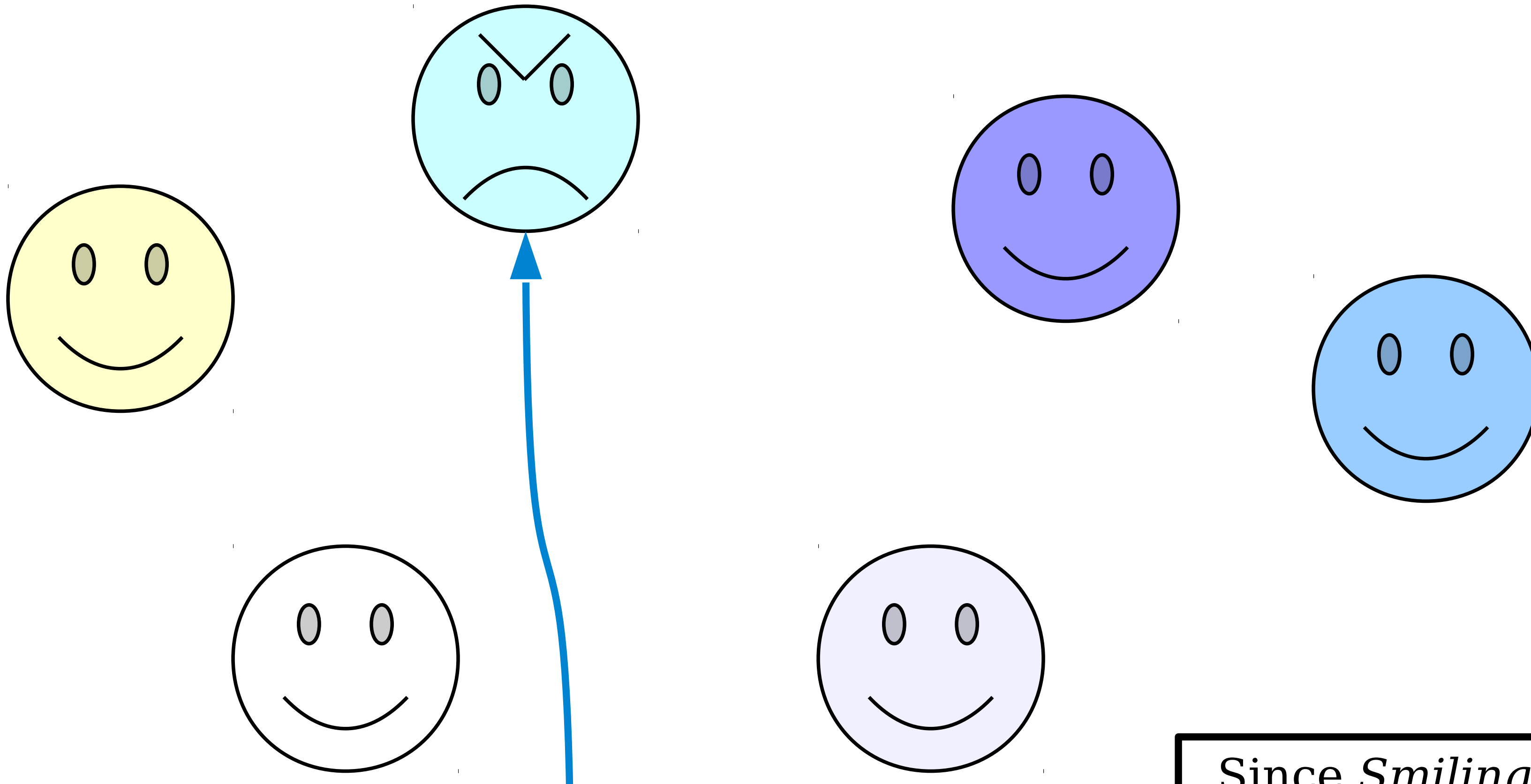
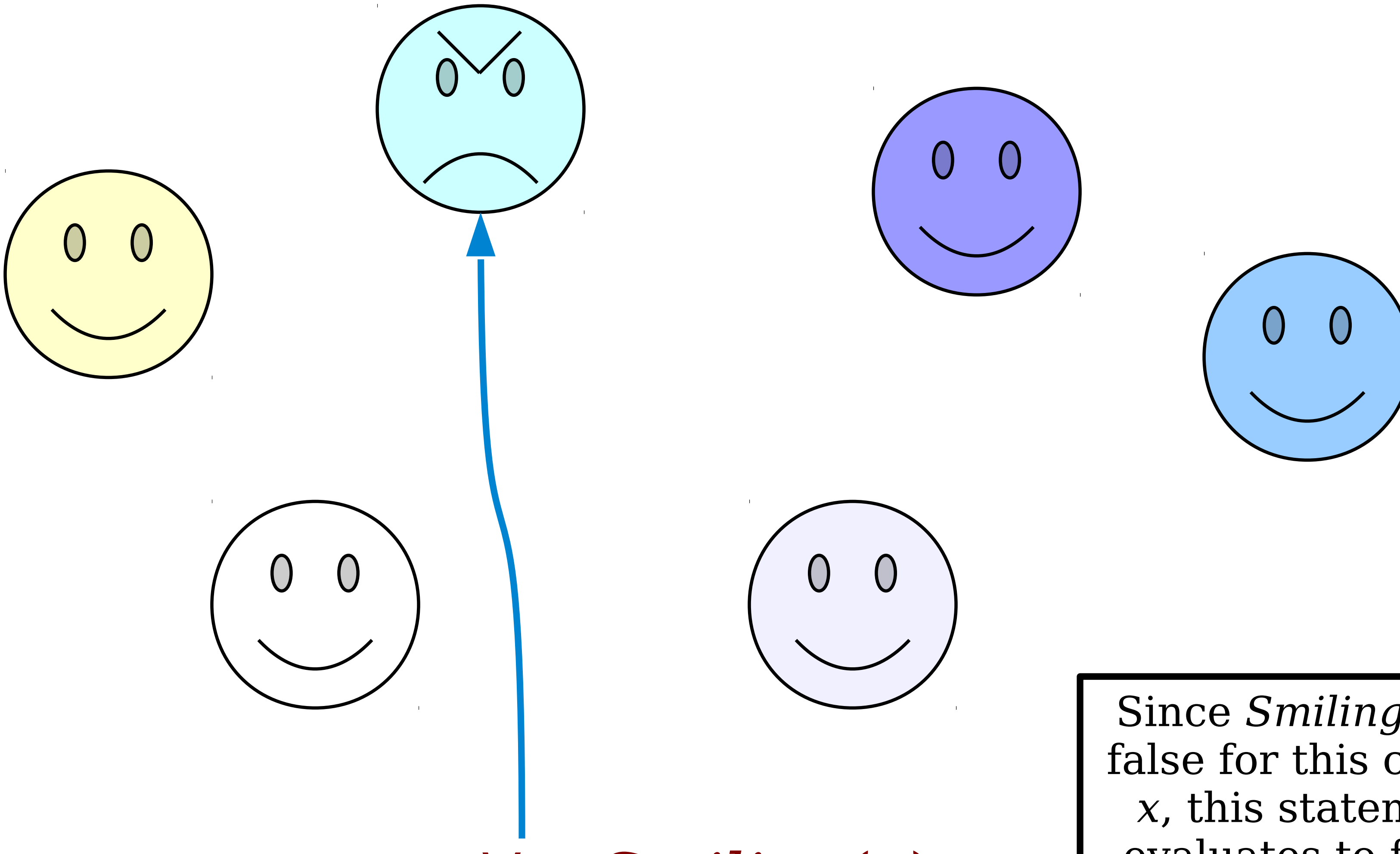


$\forall x.\ Smiling(x)$

Since $Smiling(x)$ is false for this choice x, this statement evaluates to false.

The Universal Quantifier



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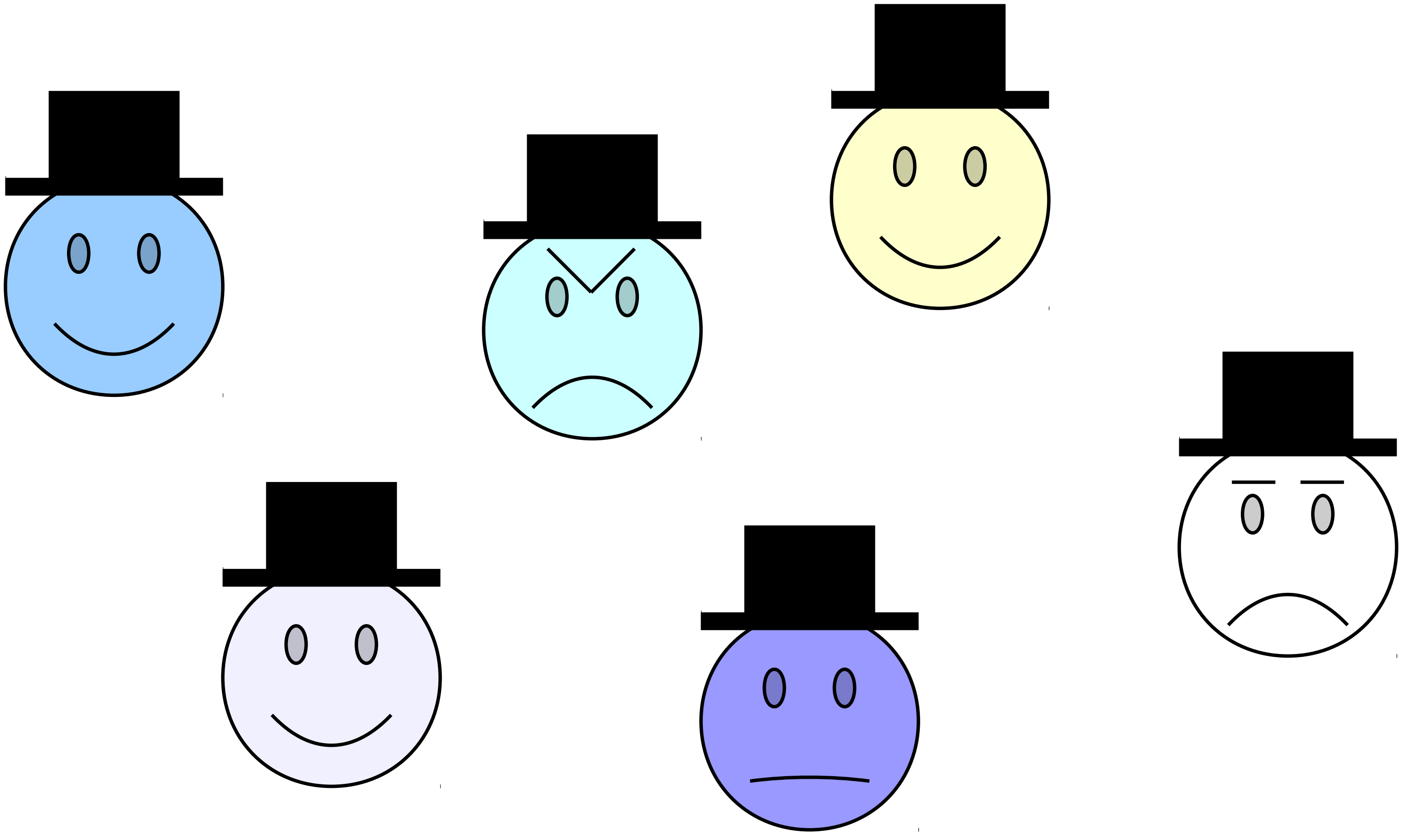
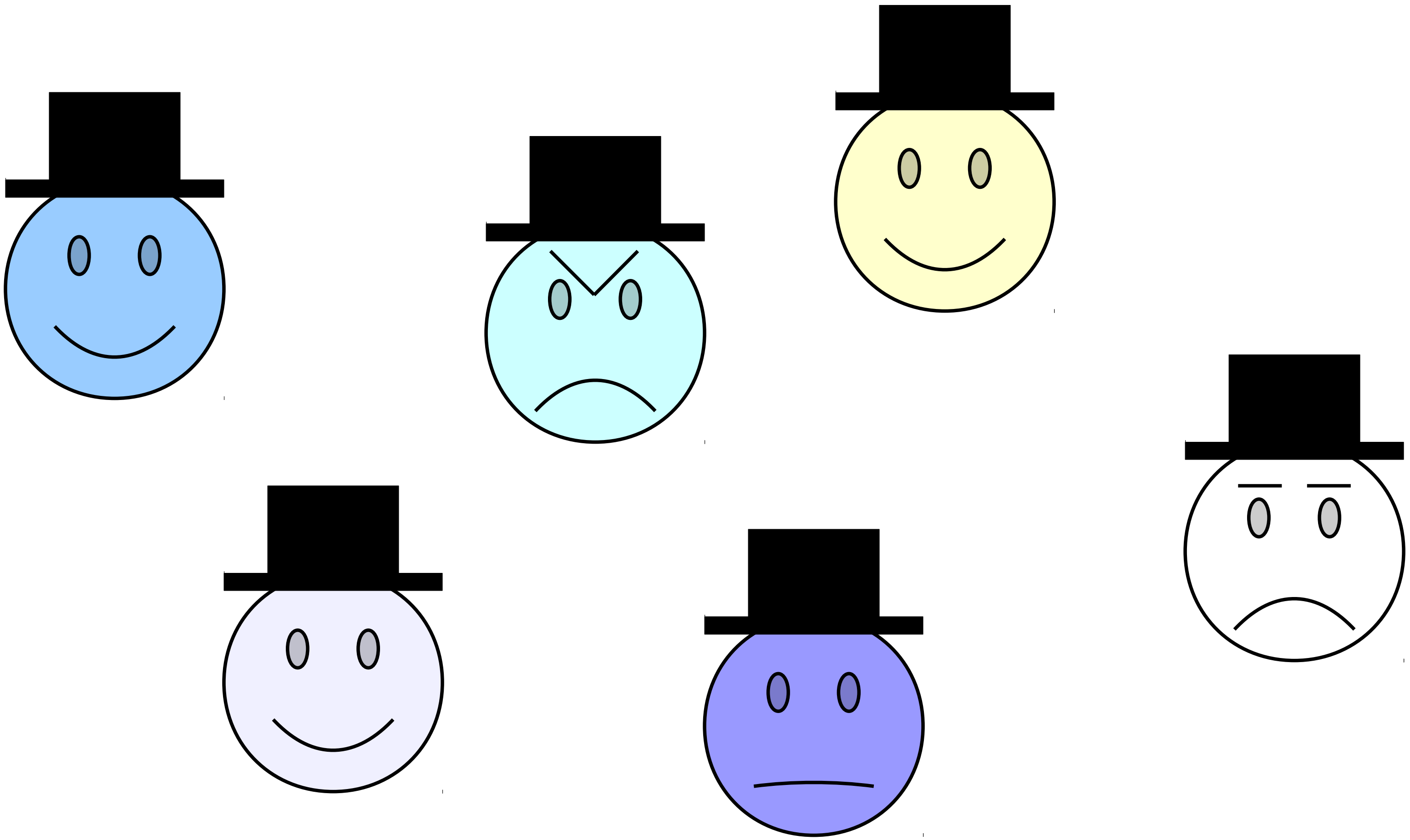


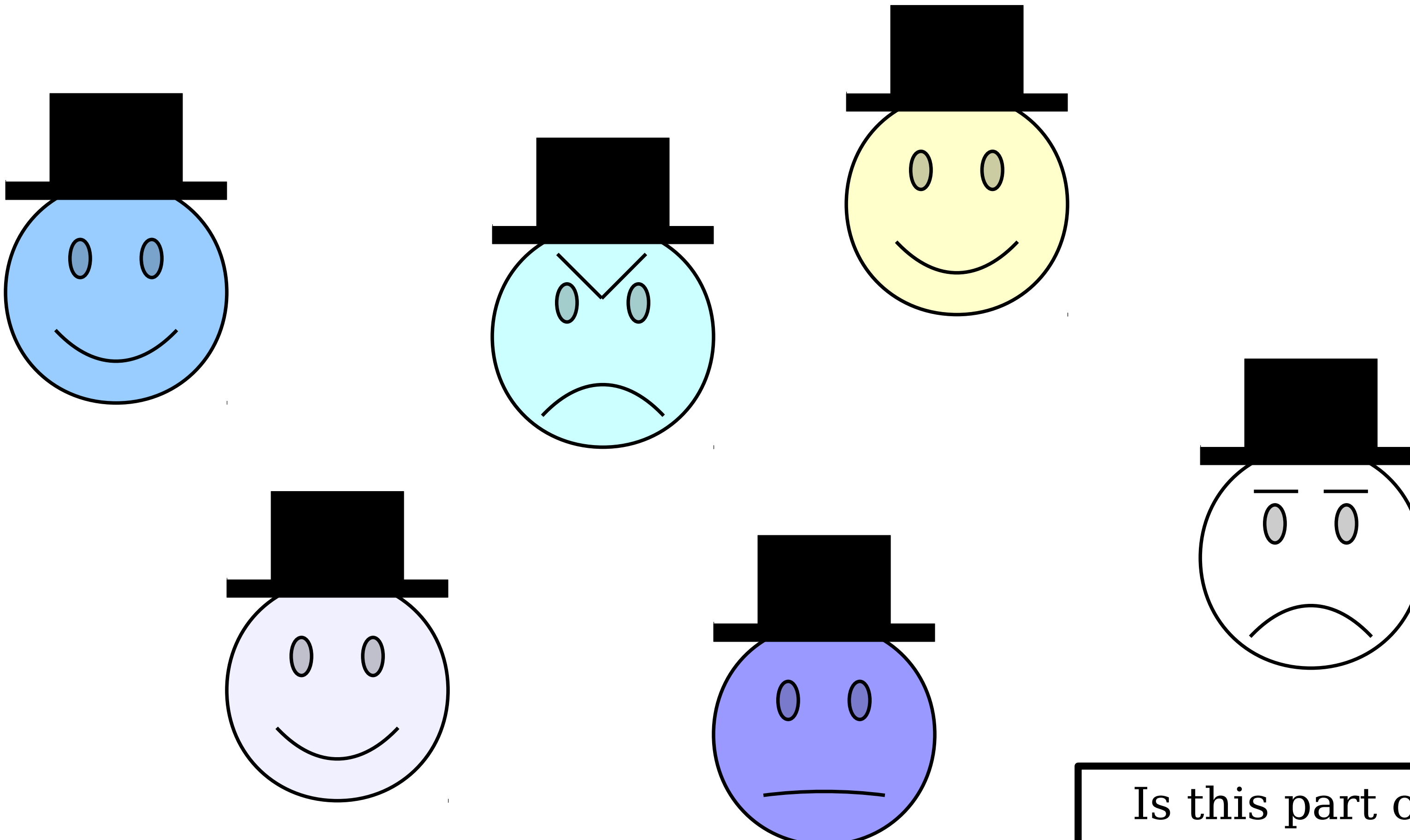
$$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$$

The Universal Quantifier



$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$

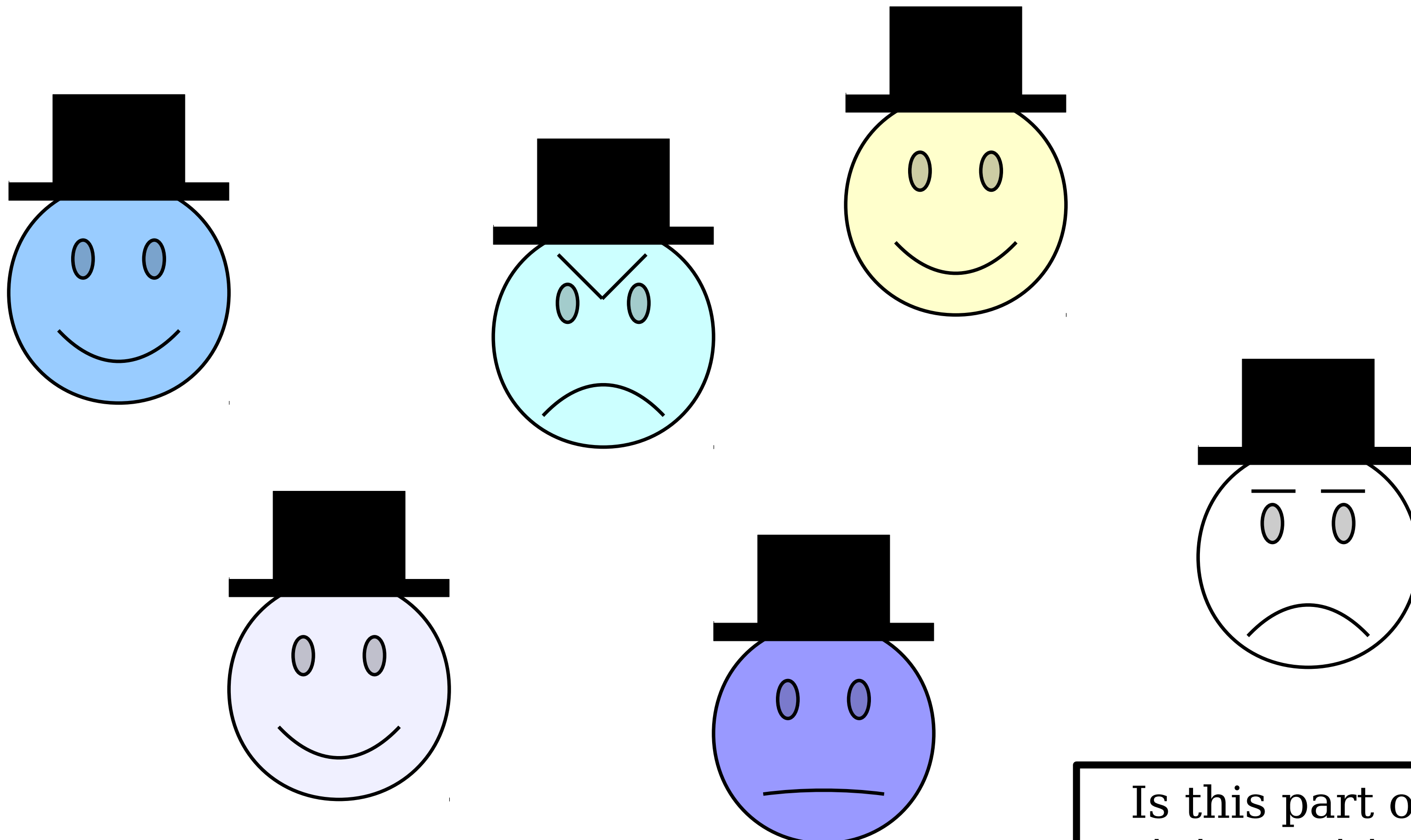
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$

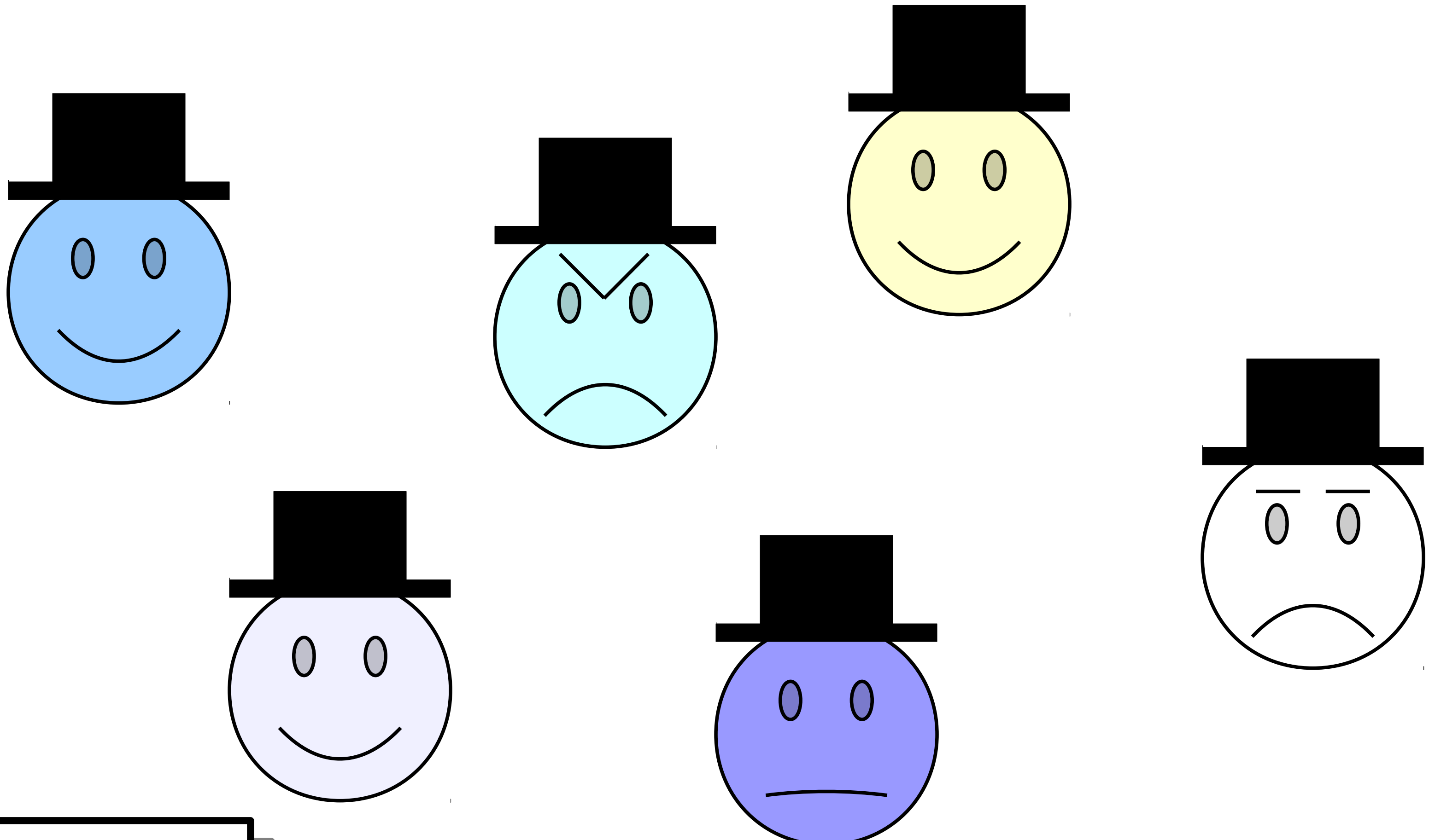
The Universal Quantifier



Is this part of the statement true or false?

$$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$$

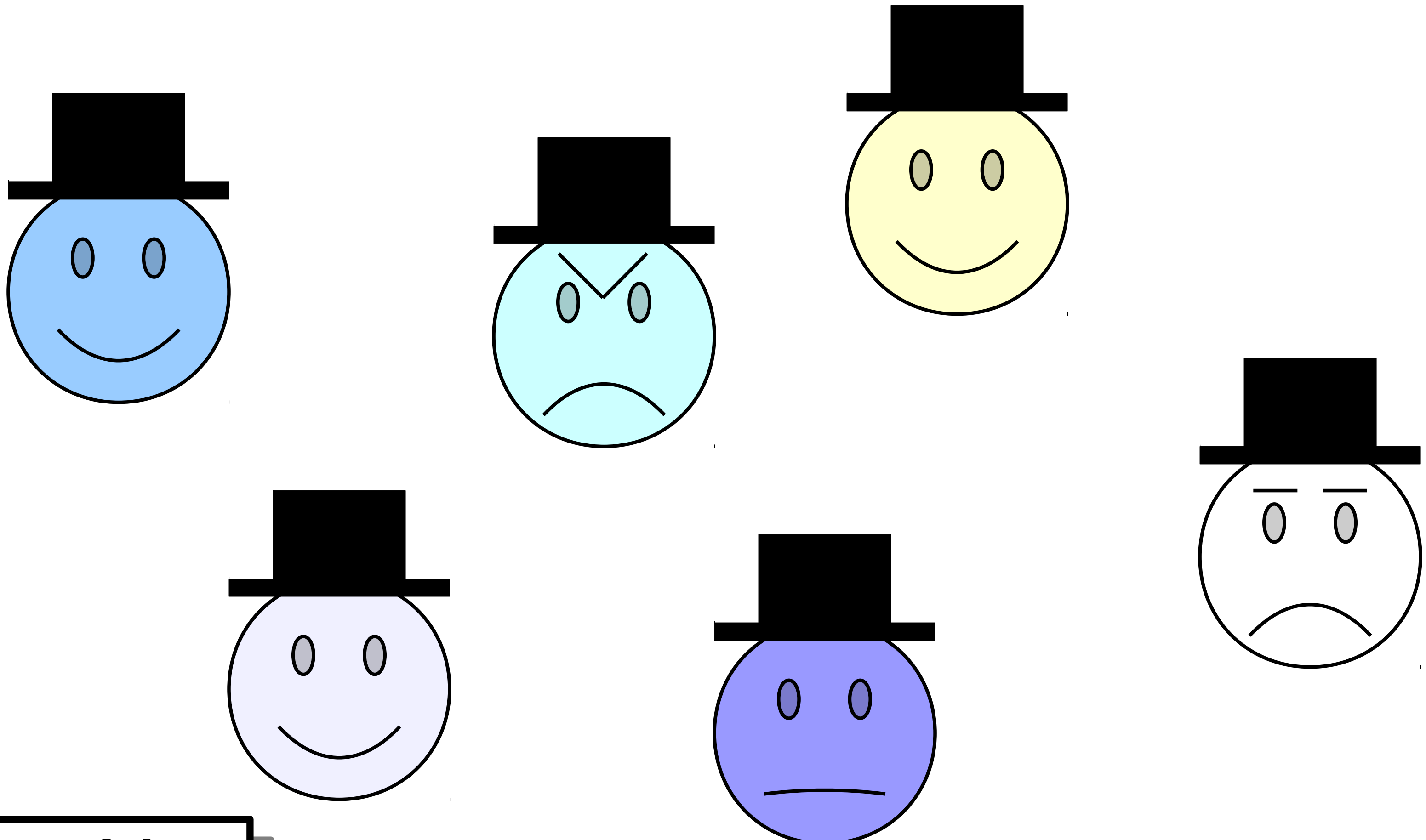
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$

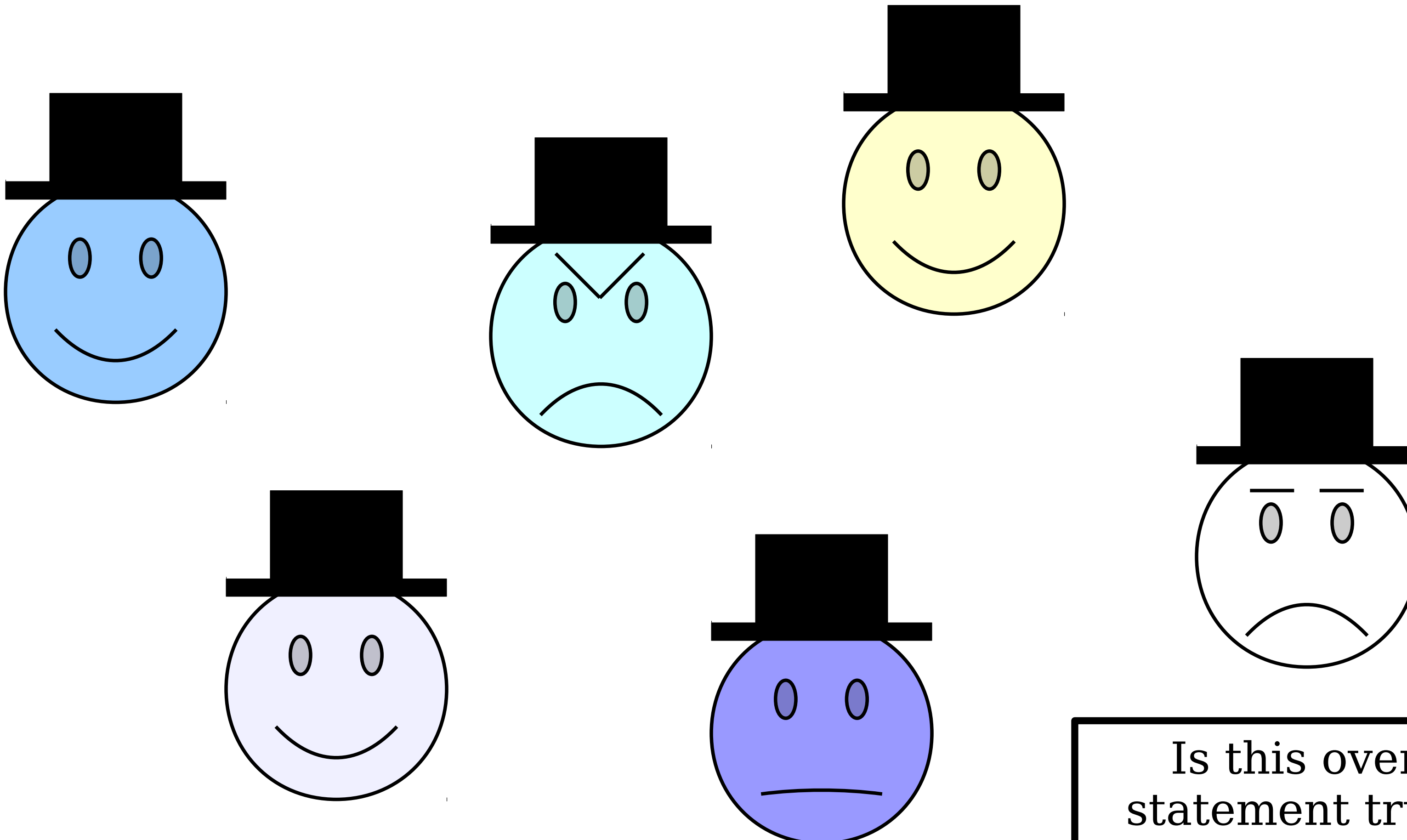
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x.\ Smiling(x))$~~ $\to (\forall y.\ WearingHat(y))$

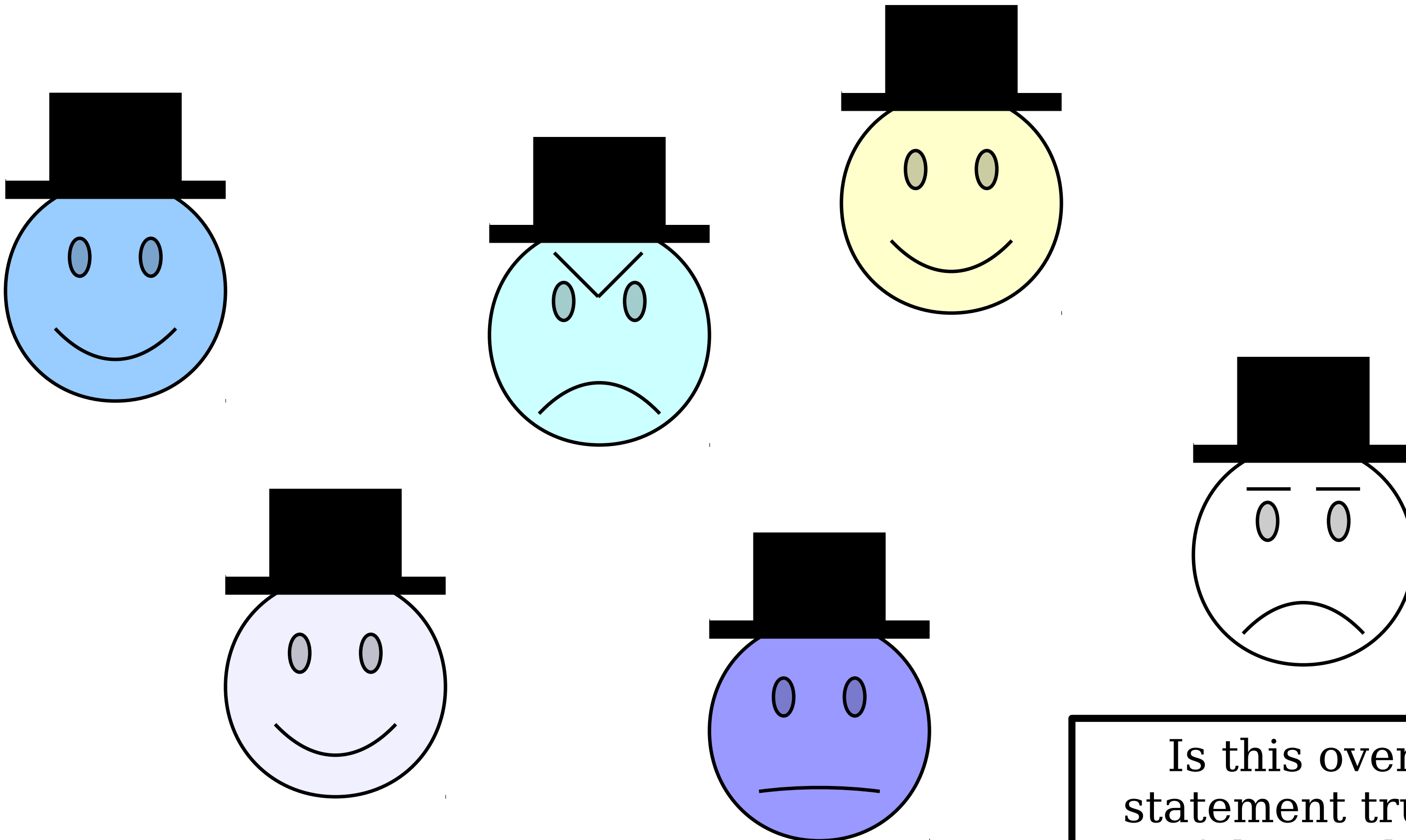
The Universal Quantifier



Is this overall statement true or false in this scenario?

~~$(\forall x.\ Smiling(x))$~~ $\rightarrow (\forall y.\ WearingHat(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$